CLOSED GEODESICS ON ORBIFOLDS

K. Guruprasad and A. Haefliger

Abstract. In this paper, we try to generalize to the case of compact Riemannian orbifolds Q some classical results about the existence of closed geodesics of positive length on compact Riemannian manifolds M. We shall also consider the problem of the existence of infinitely many geometrically distinct closed geodesics.

In the classical case the solution of those problems involve the consideration of the homotopy groups of M and the homology properties of the free loop space on M (Morse theory). Those notions have their analogue in the case of orbifolds (see [9]). The main part of this paper will be to recall those notions and to show how the classical techniques can be adapted to the case of orbifolds.

1. Summary of the results

A Riemannian orbifold structure Q on a Hausdorff topological space $|Q|$ is given by an atlas of uniformizing charts $q_i : X_i \to V_i$. The V_i, $i \in I$, are open sets whose union is $|Q|$; each X_i is a Riemannian manifold with a finite subgroup Γ_i of its group of isometries, the q_i are continuous Γ_i-invariant maps inducing a homeomorphism from $\Gamma_i \backslash X_i$ to V_i. The change of charts are Riemannian isometries.

It is convenient to assume that the X_i are disjoint and to consider on the union X of the X_i the pseudogroup $\mathcal{P}$ generated by the change of charts (its restriction to X_i is generated by the elements of Γ_i), or equivalently the topological groupoid $\mathcal{G}$ of germs of change of charts. It is an étale groupoid with space of units X, and the projection q which is the union of the q_i induces a homeomorphism from the space of orbits $\mathcal{G}\backslash X$ to $|Q|$.

The orbifold Q is said to be *developable* (*good* in the sense of Thurston [17]) if it is the quotient of a Riemannian manifold by a discrete subgroup Γ of its group of isometries.

A continuous free loop on Q is an equivalence (cohomology) class of continuous $\mathcal{G}$-cocycles (see 2.2.1) on the circle S^1. To put a topology on the set $|\Lambda^c Q|$ of continuous free loops on Q, it is convenient to represent such a free loop as an equivalence class (see 2.3) of closed $\mathcal{G}$-paths $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ over a subddivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0, 1]$. Here the c_i are continuous maps from $[t_{i-1}, t_i]$ to X, the $g_i \in \mathcal{G}$ are germs of changes of charts with source $c_{i+1}(t_i)$ and target $c_i(t_i)$, for $0 < i < k$, the source of g_0 (resp. g_k) being $c_1(0)$ (resp. the target of g_0) and the target of g_k being $c_k(1)$. The length of the free loop $[c]$ represented by c is the sum of the length of the c_i, and if the c_i are differentiable, its energy is equal to the sum of the energies of the c_i. It is a closed geodesic on Q if each

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c_i is a constant speed geodesic and the differential of g_i, for $1 < i < k$, maps the velocity vector $\dot{c}_{i+1}(t_i)$ to $\dot{c}_i(t_i)$ and the differential of $g_k g_0$ maps $\dot{c}_1(0)$ to $\dot{c}_k(1)$. Free loops of length 0 are always geodesics. They can be represented by pairs of the form (c_1, g_1), where $c_1 : [0,1] \to X$ is the constant map to a point x and g_1 an element of $\mathcal{G}$ with source and target x (well defined up to conjugation).

A continuous $\mathcal{G}$-loop based at $x \in X$ is also an equivalence class of $\mathcal{G}$-paths c as above, where x is the target of g_0 (which is egal to the source of g_k), under a more restrictive equivalence class (where the point x is preserved, see 2.3.2 for the precise definition). Let $\Omega^c_X = \Omega^c_X(\mathcal{G})$ be the set of continuous $\mathcal{G}$-loops based at the various points of X. The groupoid $\mathcal{G}$ acts naturally on Ω^c_X and there is a natural projection $q_\Lambda : \Omega^c_X \to |\Lambda^c Q|$ inducing a bijection $\mathcal{G}\backslash\Omega^c_X \to |\Lambda^c Q|$.

To each orbifold Q is associated a classifying space BQ (see [9]). Roughly speaking, it is a space BQ with a projection $\pi : BQ \to |Q|$ such that, for each $y \in |Q|$, the fiber $\pi^{-1}(y)$ is a space whose universal covering in contractible and whose fundamental group is isomorphic to the isotropy subgroup of a point $x \in q_i^{-1}(y)$ in a uniformizing chart $q_i : X_i \to V_i \subseteq |Q|$. The elements of $|\Lambda^c Q|$ are in bijection with equivalence classes of continuous maps $\mathbb{S}^1 \to BQ$, two maps being equivalent if they are homotopic through an homotopy projecting under π to a constant homotopy. The fundamental group of BQ is isomorphic to the orbifold fondamental group of Q.

For the proof of the following proposition, see 3.1 and 3.2.

Proposition. Ω^c_X *is naturally a Banach manifold. The action of* $\mathcal{G}$ *is differentiable and the quotient space* $|\Lambda^c Q|$ *has a natural structure of Banach orbifold noted* $\Lambda^c Q$.

If BQ *and* $B\Lambda^c Q$ *denote the classifying spaces of the orbifolds* Q *and* $\Lambda^c Q$*, then* $B\Lambda^c Q$ *has the same weak homotopy type as the topological space* ΛBQ *of continuous free loops on* BQ.

The fact that the free loop space of an orbifold has an orbifold structure was also observed independently by Weimin Chen [3].

Let $|\Lambda Q|$ (resp. Ω_X) be the sets of $\mathcal{G}$-loops (resp. based $\mathcal{G}$-loops) of class H^1, namely those loops represented by closed $\mathcal{G}$-paths $c = (c_1, g_1, \ldots, c_k, g_k)$ such that each path c_i is absolutely continuous and the velocity $|\dot{c}_i|$ is square integrable. The energy $E(c)$ of c is equal to the sum of the energies of the c_i and depends only on its equivalence class. The next proposition is proved in 3.3.2 and 3.3.5.

Proposition. Ω_X *is naturally a Riemannian manifold and* $|\Lambda Q|$ *has a Riemannian orbifold structure noted* ΛQ.

The natural inclusion $\Lambda Q \to \Lambda^c Q$ *induces a homotopy equivalence on the corresponding classifying spaces.*

One proves like in the classical case that the energy function E is differentiable on ΛQ and that its critical points correspond to the closed geodesics. If the orbifold Q is compact, the Palais-Smale condition holds. The usual techniques (Lusternik-Schnierelmann, Morse) as explained for instance in Klingenberg [11] or [12], can then be applied to study the existence of closed geodesics on compact Riemannian orbifolds. We get in particular the following results, proved in 5.1.

Theorem. *Let* Q *be a compact Riemannian orbifold. There is a closed geodesic on* Q *of positive length in each of the following cases:*

1) Q *is not developable.*

2) The (orbifold) fundamental group of Q has an element of infinite order or is finite.

We don't know if this covers all the cases. It would be so if a finitely presented torsion group would be finite (see the remark 5.1.2).

We say that two closed geodesics on Q are geometrically distinct if their projections to $|Q|$ have distinct images. The following result (see 5.2.3) is an extension to orbifolds of a classical result of Gromoll-Meyer [5] and Vigué-Sullivan [18].

Theorem. *Let Q be a compact simply connected Riemannian orbifold such that the rational cohomology of $|Q|$ is not generated by a single element. Then there exist on Q an infinity of geometrically distinct closed geodesics of positive length.*

The paper is organized as follows. The lengthy section 2 recalls elementary and classical definitions concerning orbifolds, "maps" from spaces to orbifolds, in particular closed curves (or free loops) on orbifolds, classifying spaces of orbifolds, etc. In section 3 we explain the orbifold structure on the space $|\Lambda^c Q|$ of continuous free loops and on the space $|\Lambda Q|$ of free loops of class H^1 on Q and the basic properties of the energy function. In section 4 we study the orbifold tubular neighbourhoods of the $\mathbb{S}^1$-orbits of closed geodesics in a geometric equivalence class. In the last section, we sketch how the classical theory of closed geodesics on Riemannian manifolds can be adapted to the case of orbifolds. In sections 3, 4 and 5 we assume familiarity with the notions and the basic papers concerning the theory of closed geodesics in classical Riemannian geometry.

A first version of this paper was submitted to Topology. In his report, the referee pointed out to us the rich literature (we were not aware of) concerning the theory of invariant closed geodesics, in particular the work of Grove-Tanaka [6]. Thanks to his remarks we were able in this revised version to extend to compact simply connected Riemannian orbifold the result of Gromoll-Meyer [5], using [6]. The interpretation in our framework of the results of Grove-Tanaka and Grove-Halperin [7] is mentionned in 3.3.6 and 5.2.5.

2. Basic definitions

In this section we recall basic definitions and known results (see [9] and [2]).

2.1 Orbifolds

2.1.1. Definition of an orbifold structure. Let $|Q|$ be a Hausdorff topological space. A differentiable orbifold structure Q on $|Q|$ is given by the following data:

i) an open cover $\{V_i\}_{i\in I}$ of $|Q|$ indexed by a set I,

ii) for each $i \in I$, a finite subgroup Γ_i of the group of diffeomorphisms of a connected differentiable manifold X_i and a continuous map $q_i : X_i \to V_i$, called a uniformizing chart, inducing a homeomorphism from $\Gamma_i \backslash X_i$ onto V_i,

iii) for each $x_i \in X_i$ and $x_j \in X_j$ such that $q_i(x_i) = q_j(x_j)$, there is a diffeomorphim h from an open connected neighbourhood W of x_i to a neighbourhood of x_j such that $q_j \circ h = q_{i|W}$. Such a map h is called a *change of chart*; it is well defined up to composition with an element of Γ_j; if $i = j$, then h is the restriction of an element of Γ_i.

The family (X_i, q_i) is called an *atlas of uniformizing charts* defining the orbifold

$|Q|$ if, put together, they satisfy the compatibility condition iii). It is easy to check that the above definition of orbifolds is equivalent to the definition of V-varieties introduced by Satake (as explained in [10]) and to the definition of Thurston [17].

A *Riemannian orbifold* is an orbifold Q defined by an atlas of uniformizing charts such that the X_i are Riemannian manifolds and the change of charts are Riemannian isometries. Note that on any paracompact differentiable orbifold one can introduce a Riemannian metric.

2.1.2. The pseudogroup of change of charts. Developability. Let X be the disjoint union of the X_i. We identify each X_i to a connected component of X and we note $q : X \to |Q|$ the union of the maps q_i. Any diffeomorphim h from an open subset U of X to an open subset of X such that $q \circ h = q_{|U}$ will be called a change of charts. The collection of change of charts form a pseudogroup $\mathcal{P}$ of local diffeomorphisms of X, called the pseudogroup of change of charts of the orbifold (with respect to the uniformizing atlas (X_i, q_i)). Two points $x, y \in X$ are said to be in the same orbit of $\mathcal{P}$ if there is an element $h \in \mathcal{P}$ such that $h(x) = y$. This defines an equivalence relation on X whose classes are called the orbits of $\mathcal{P}$. The quotient of X by this equivalence relation, with the quotient topology, will be denoted $\mathcal{P}\backslash X$. The map $q : X \to |Q|$ induces a homeomorphism from $\mathcal{P}\backslash X$ to $|Q|$.

The pseudogroups of change of charts of two atlases defining the same orbifold structure on $|Q|$ are equivalent in the following sense. Two pseudogroups $\mathcal{P}_0$ and $\mathcal{P}_1$ of local diffeomorphisms of differentiable manifolds X_0 and X_1 respectively are equivalent if there is a pseudogroup $\mathcal{P}$ of local diffeomorphims of the disjoint union X of X_0 and X_1 whose restriction to X_j is equal to $\mathcal{P}_j$ and such that the inclusion of X_j into X induces a homeomorphism $\mathcal{P}_j\backslash X_j \to \mathcal{P}\backslash X$, $j = 0, 1$.

More generally, consider a pseudogroup of local diffeomorphisms $\mathcal{P}$ of a differentiable manifold X such that each point x of X has an open neighbourhood U such that the restriction of $\mathcal{P}$ to U is generated by a finite group Γ_U of diffeomorphisms of U. Assume moreover that the space of orbits $\mathcal{P}\backslash X$ is Hausdorff. Then $\mathcal{P}\backslash X$ has a natural orbifold structure still noted $\mathcal{P}\backslash X$.

For instance if Γ is a discrete subgroup of the group of diffeomorphisms of a manifold X whose action on X is proper, then $\Gamma\backslash X$ has a natural orbifold structure. An orbifold structure arising in this way is called *developable*. In this case, the pseudogroup of change of charts is generated by the elements of Γ.

2.1.3. The tear drop (a non-developable orbifold) [17]**.** The topological space $|Q|$ is the 2-sphere $\mathbb{S}^2$ with north pole N and south pole S. The north pole is a conical point of order n, all the other points are regular. The orbifold structure can be defined by an atlas of two uniformizing charts $q_i : X_i \to V_i$, $i = 1, 2$. Here X_i is the open disc of radius $3\pi/4$ centered at 0 in $\mathbb{R}^2$ with polar coodinates (r, θ), the group Γ_1 is generated by a rotation of order n and Γ_2 is trivial. The map q_1 (resp. q_2) maps the point (r, θ) to the point of $\mathbb{S}^2$ with geodesic coordinates $(r, n\theta)$ (resp. (r, θ)) centered at the north pole (resp. the south pole). The map $q_2^{-1}q_1$ is the n-fold covering $(r, \theta) \mapsto (\pi - r, n\theta)$ of the annulus $A := \{(r, \theta), \pi/4 < r < 3\pi/4\}$. On the disjoint union $X = (\{1\}\times X_1) \cup (\{2\}\times X_2)$ of X_1 and X_2, the pseudogroup of changes of charts is generated by Γ_1 and by the diffeomorphisms from open sets of $\{1\} \times A$ to open sets of $\{2\} \times A$ which are restrictions of the map $(1, a) \mapsto (2, q_2^{-1}q_1(a))$. Any Riemannian metric on Q is given by a Riemannian metric on X_1 invariant by Γ_1 and a Riemannian metric on X_2 such that $q_2^{-1}q_1$ is a local isometry.

2.1.4. The étale groupoid of germs of change of charts. Recall that a groupoid $(\mathcal{G}, X)$ is a small category $\mathcal{G}$ with set of objects X, all elements of $\mathcal{G}$ being invertible. The set of objects X is often identified to the set of units of $\mathcal{G}$ by the map associating to an object $x \in X$ the unit $1_x \in \mathcal{G}$. Each element $g \in \mathcal{G}$ is considered as an arrow with source $\alpha(g) \in X$ (identified to its right unit) and target $\omega(g) \in X$ (identified to its left unit). The inverse of g is denoted g^{-1}. For $x \in \mathcal{G}$, the elements g of $\mathcal{G}$ with $\alpha(g) = \omega(g)$ form a group noted $\mathcal{G}_x$, called the isotropy subgroup of x.

A topological groupoid is a groupoid $(\mathcal{G}, X)$ such that $\mathcal{G}$ and X are topological spaces, all the structure maps (projections α, $\omega : \mathcal{G} \to X$, composition, passage to inverse) are continuous, and such that the map $x \mapsto 1_x$ from X to $\mathcal{G}$ is a homeomorphism onto its image. An *étale groupoid* is a topological groupoid such that the projections α and ω from $\mathcal{G}$ to X are étale, i.e. are local homeomorphisms.

To the pseudogroup $\mathcal{P}$ of change of charts of an atlas of uniformizing charts defining an orbifold, we can associate the étale groupoid $(\mathcal{G}, X)$ of all germs of change of charts, with the usual topology of germs, X being the disjoint union of the sources of the charts. From $\mathcal{G}$ we can reconstruct $\mathcal{P}$, because its elements can be obtained as the diffeomorphisms from open sets U of X to open sets of X which are the composition with ω of sections $U \to \mathcal{G}$ of α above U. We shall also use the notation $Q = \mathcal{G}\backslash X$ to denote an orbifold whose pseudogroup $\mathcal{P}$ of change of charts is equivalent to the pseudogroup corresponding to $\mathcal{G}$.

For instance, consider the case where Q is developable, quotient of a smooth manifold X by a discrete group Γ of diffeomorphisms of X, acting properly on X. Then $\mathcal{G}$ is the groupoid $\Gamma \times X$, where Γ is endoved with the discrete topology, the projection α (resp. ω) mapping (γ, x) to x (resp. $\gamma.x$). The composition $(\gamma, x)(\gamma', x')$ is defined when $x = \gamma'.x'$ and is equal to $(\gamma\gamma', x')$. We shall use the notation $\mathcal{G} = \Gamma \ltimes X$ and the orbifold $Q = \Gamma\backslash X$ is also noted $\mathcal{G}\backslash X$.

The following lemma will be useful later on.

2.1.5. Lemma. *Assume that $Q = \mathcal{G}\backslash X$ is a Riemannian orbifold, and let $g \in \mathcal{G}$ with $x = \alpha(g)$ and $y = \omega(g)$. Let $B(x, \epsilon)$ and $B(y, \epsilon)$ be two convex geodesic balls centered at x and y with radius ϵ (such balls always exist for small enough ϵ). Then there is a unique element h of the pseudogroup of changes of charts $\mathcal{P}$ which is an isometry from $B(x, \epsilon)$ to $B(y, \epsilon)$ and whose germ at x is equal to g.*

Proof. By hypothesis the exponential map is defined on the ball of radius ϵ centered at the origin of T_xX and is a diffeomorphism of that ball to $B(x, \epsilon)$; similarly for y. Therefore let $h : B(x, \epsilon) \to B(y, \epsilon)$ be the diffeomorphism mapping isometrically geodesic rays issuing from x to those issuing from y and whose differential at x is the differential Dg of a representative of g. For $0 < r < \epsilon$, assume that the restriction h_r of h to the open ball $B(x, r)$ belongs to $\mathcal{P}$; this is the case if r is small enough. It will be sufficient to prove that the restriction of h to a neighbourhood of the closure of $B(x, r)$ belongs to $\mathcal{P}$. For a point $z \in \partial B(x, r)$, the points z and $h(z)$ are in the same orbit under $\mathcal{P}$, because $|Q|$ is Hausdorff. There are small ball neighbourhoods U of z and V of $h(z)$ and an element $f : U \to V$ of $\mathcal{P}$ such that the restriction of $\mathcal{P}$ to U is generated by a group Γ_U of diffeomorphisms of U and such that the germ of any element of $\mathcal{P}$ with source in U and target in V is the germ of the composition of f with an element of Γ_U. Therefore, the restriction of h_r to $B(x, r) \cap U$ is the restriction of an element of $\mathcal{P}$ defined on U. As such an element is a Riemannian isometry, it must coincide with $h_{|U}$ on the geodesic rays issuing from x, hence on U. $\square$

2.2 Morphisms from spaces to topological groupoids

2.2.1. Definition using cocycles [8]. Let $\mathcal{G}$ be a topological groupoid with space of units X, source and target projections $\alpha, \omega : \mathcal{G} \to X$ respectively. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of a topological space K. A 1-cocycle over $\mathcal{U}$ with value in $\mathcal{G}$ is a collection of continuous maps $f_{ij} : U_i \cap U_j \to \mathcal{G}$ such that, for each $x \in U_i \cap U_j \cap U_k$, we have:

$$f_{ik}(x) = f_{ij}(x) f_{jk}(x).$$

This implies in particular that $f_{ii}(x)$ is a unit of $\mathcal{G}$ and that $f_i := f_{ii}$ can be considered as a continuous map from U_i to X. Also $f_{ij} = f_{ji}^{-1}$.

Two cocycles on two open covers of K with value in $\mathcal{G}$ are equivalent if there is a cocycle with value in $\mathcal{G}$ on the disjoint union of those two covers extending the given ones on each of them. An equivalence class of cocycles is called a (continuous) morphism from K to $\mathcal{G}$ (or when Q is an orbifold $\mathcal{G} \backslash X$ a "continuous map" from K to Q). The set of equivalence classes of 1-cocycles on K with value in $\mathcal{G}$ is noted $H^1(K, \mathcal{G})$. One should observe that *if $\mathcal{G}$ and $\mathcal{G}'$ are the groupoids of germs of the changes of charts of two atlases defining the same orbifold structure on a space $|Q|$, then there is a natural bijection bvetween the sets $H^1(K, \mathcal{G})$ and $H^1(K, \mathcal{G}')$.*

Any continuous map f from a topological space K' to K induces a map $f^* : H^1(K, \mathcal{G}) \to H^1(K', \mathcal{G})$. Two morphisms from K to $\mathcal{G}$ are homotopic if there is a morphism from $K \times [0,1]$ to $\mathcal{G}$ such that the morphisms from K to $\mathcal{G}$ induced by the natural inclusions $k \mapsto (k, i)$, $i = 0, 1$, from K to $K \times [0,1]$ are the given morphisms.

Any morphism from K to $\mathcal{G}$ projects, via $q : X \to \mathcal{G} \backslash X = |Q|$, to a continuous map from K to $|Q|$; note that two distinct morphisms may have the same projection (see the example at the end of 2.3.5).

2.2.2. Principal $\mathcal{G}$-bundles. Another description of morphisms from K to $\mathcal{G}$ can be given in terms of isomorphism classes of principal $\mathcal{G}$-bundles over K (see [9]).

Let E be a topological space with a continuous map $\alpha_E : E \to X$. Let $E \times_X \mathcal{G}$ be the subspace of $E \times \mathcal{G}$ consisting of pairs (e, g) such that $\alpha_E(e) = \omega(g)$. A continuous (right) action of $\mathcal{G}$ on E with respect to α_E is a continuous map $(e, g) \mapsto e.g$ from the space $E \times_X \mathcal{G}$ to E such that $\alpha(g) = \alpha_E(e.g)$, $(e.g).g' = e.(gg')$ and $e.1_x = e$. Left actions are defined similarly.

A principal $\mathcal{G}$-bundle over K is a topological space E together with a surjective continuous map $p_E : E \to K$, called the bundle projection, and a continuous action $(e, g) \mapsto e.g$ of $\mathcal{G}$ on E with respect to a continuous map $\alpha_E : E \to X$, called the action map, such that $p(e.g) = p(e)$. Moreover we assume that the action is simply transitive on the fibers of p in the following sense. Each point of K has an open neighbourhood U with a continuous section $s : U \to E$ with respect to p_E such that the map $U \times_X \mathcal{G} \to p^{-1}(U)$ mapping pairs $(u, g) \in U \times \mathcal{G}$ with $\omega(g) = \alpha_E s(u)$ to $s(u).g$ is a homeomorphism. It follows that if $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of K and if $s_i : U_i \to E$ is a local continuous section of p above U_i for each $i \in I$, then there are unique continuous maps $f_{ij} : U_i \cap U_j \to \mathcal{G}$ such that $s_i(u) = s_j(u) f_{ji}(u)$ for each $u \in U_i \cap U_j$. Thus $f = (f_{ij})$ is a 1-cocycle over $\mathcal{U}$ with value in $\mathcal{G}$.

Conversely, if $f = (f_{ij})$ is a 1-cocycle over an open cover $\mathcal{U} = (U_i)_{i \in I}$ of K with value in $\mathcal{G}$, then we can construct a principal $\mathcal{G}$-bundle E over K by identifying in the disjoint union of the $U_i \times_X \mathcal{G} = \{(u, g) \in U_i \times \mathcal{G} : \omega(g) = f_i(u)\}$ the point

$(u, g) \in U_i \times_X \mathcal{G}$, $u \in U_i \cap U_j$, with the point $(u, f_{ji}(u)g) \in U_j \times_X \mathcal{G}$. The projections $p_E : E \to K$ and $\alpha_E : E \to X$ map the equivalence class of $(u, g) \in U_i \times_X \mathcal{G}$ to u and $\alpha(g)$ resp. and the action of g' on the class of (u, g) is the class of (u, gg'). A principal $\mathcal{G}$-bundle obtained in this way by using an equivalent cocycle is isomorphic to the preceding one, i.e. there is a homeomorphism between them projecting to the identity of K and commuting with the action of $\mathcal{G}$. This isomorphism is determined uniquely by a cocycle extending the two given cocycles.

Therefore we see that *there is a natural bijection between the set* $H^1(K, \mathcal{G})$ *and the set of isomorphism classes of principal* $\mathcal{G}$*-bundles over* K. This correspondence is functorial via pull back: if E is a principal $\mathcal{G}$-bundle over K and if $f : K' \to K$ is a continuous map, then the pull back f^*E of E by f (or the bundle induced from E by f) is the bundle $K' \times_K E$ whose elements are the pairs $(k', e) \in K' \times E$ such that $f(k')p_E(e)$. The projection (resp. the action map) sends (k', e) to k' (resp. to $\alpha_E(e)$

$\mathcal{G}$ itself can be considered as a principal $\mathcal{G}$-bundle over X with respect to the projection $\omega : \mathcal{G} \to X$, the action map $\alpha_{\mathcal{G}} : \mathcal{G} \to X$ being the source projection. Any principal $\mathcal{G}$-bundle E over K is locally the pull back of this bundle $\mathcal{G}$ by a continuous map to X. The projection p_E is an étale map (i.e locally a homeomorphism) when $\mathcal{G}$ is an étale groupoid.

2.2.3. Relative morphisms. Let K be a topological space, $L \subseteq K$ be a subspace and F be a principal $\mathcal{G}$-bundle over L. A morphism from K to $\mathcal{G}$ relative to F is represented by a pair (E, ϕ) where E is a principal $\mathcal{G}$-bundles E over K and ϕ is an isomorphism from F to the restriction $E_{|L}$ of E above L. Two such pairs (E, ϕ) and (E', ϕ') represent the same morphism from K to $\mathcal{G}$ relative to F if there is an isomorphism $\Phi : E \to E'$ such that $\phi' = \Phi \circ \phi$.

An important particular case is when L is a base point $z \in K$ and F is the pull back of $\mathcal{G}$ by the map sending z to a given point $x \in X$, i.e. F is the set of pairs (z, g), where g is an element of $\mathcal{G}$ with target x. A morphism from K to $\mathcal{G}$ relative to F (in that case we shall say a morphism from K to $\mathcal{G}$ mapping $z \in K$ to $x \in X$) is just represented by a $\mathcal{G}$-bundle E over K with a base point $e \in E$ with $p_E(e) = z$ and $\alpha_E(e) = x$. Indeed an isomorphism from F to $p_E^{-1}(z)$ is determined by the image e of $(z, 1_x) \in F$. Two such pairs (E, e) and (E', e') represent the same morphism if there is an isomorphism from E to E' mapping e to e' and projecting to the identity of K. More generally, suppose that F is the pull back of $\mathcal{G}$ by a continuous map $f_0 : L \to X$. Then a morphism from K to $\mathcal{G}$ relative to F (we shall say relative to f_0) is given by a principal $\mathcal{G}$-bundle E over K with a continuous map $s : L \to E$ with $p_E \circ s$ the identity of L and $\alpha_E \circ s = f_0$.

Two morphisms represented by (E_0, ϕ_0) and (E_1, ϕ_1) from K to $\mathcal{G}$ relative to F are homotopic (relative to F) if there is a bundle E over $K \times I$ and an isomorphism from $E_{|(K \times \partial I) \cup (L \times I)}$ to the bundle obtained by gluing $F \times I$ to $E_0 \times \{0\}$ and $E_1 \times \{1\}$ using the isomorphims ϕ_0 and ϕ_1.

We leave to the reader a description of isomorphism classes of relative bundles in terms of equivalence classes of relative cocycles.

Let $I^n = [0, 1]^n$ be the n-cube, and let ∂I^n be its boundary. Fix a base point x in X. Let f_0 be the constant map from ∂I^n to X. We define $\pi_n((\mathcal{G}, X), x)$ as the set of homotopy classes of principal $\mathcal{G}$-bundle over I^n relative to f_0. One proves as usual that this set has a natural group structure, called the n^{th}- homotopy group of $(\mathcal{G}, X)$ based at x. In the case where $\mathcal{G} \backslash X$ is a connected orbifold Q, this group

is called the n-th homotopy group of Q, and for n=1 the (orbifold) fundamental group of Q.

2.3 Paths and loops in orbifolds

In this section we describe in a more concrete way the morphisms from the interval $I = [0,1]$ to a topological groupoid $(\mathcal{G}, X)$ relative to a map from ∂I to X. Whe shall deal with an orbifold struture Q on a topological space $|Q|$ defined by an atlas of uniformizing charts, $(\mathcal{G}, X)$ will be the étale topological groupoid of germs of change of charts, or more generally the groupoid of germs of elements of a pseudogroup defining Q (cf. 2.1.2 and 2.1.4). As above, we have the map $q : X \to |Q|$ inducing a homeomorphism from the space of orbits $\mathcal{G}\backslash X$ to $|Q|$.

2.3.1. Continuous $\mathcal{G}$-paths. Let x and y be two points of X. A (continuous) $\mathcal{G}$-path from x to y over a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0,1]$ is a sequence $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ where

i) $c_i : [t_{i-1}, t_i] \to X$ is a continuous map,

ii) g_i is an element of $\mathcal{G}$ such that $\alpha(g_i) = c_{i+1}(t_i)$ for $i = 0, 1, \ldots, k-1$, $\omega(g_i) = c_i(t_i)$ for $i = 1, \ldots, k$, and $\omega(g_0) = x$, $\alpha(g_k) = y$.

2.3.2. Equivalence classes of $\mathcal{G}$-paths. The set $\Omega^c_{x,y}$. Among $\mathcal{G}$-paths from x to y parametrized by $[0,1]$ we define an equivalence relation generated by the following two operations:

i) Given a $\mathcal{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ over the subdivision $0 = t_0 < \cdots < t_k = 1$, we can add a subdivision point $t' \in (t_{i-1}, t_i)$ together with the unit element $g' = 1_{c_i(t')}$ to get a new sequence, replacing c_i in c by c'_i, g', c''_i, where c'_i and c''_i are the restriction of c_i to the intervals $[t_{i-1}, t']$ and $[t', t_i]$.

ii) Replace the $\mathcal{G}$-path c by a new path $c' = (g'_0, c'_1, g'_1, \ldots, c'_k, g'_k)$ over the same subdivision as follows: for each $i = 1, \ldots, k$, choose continuous maps $h_i : [t_{i-1}, t_i] \to \mathcal{G}$ such that $\alpha(h_i(t)) = c_i(t)$, and define $c'_i : t \mapsto \omega(h_i(t))$, $g'_i = h_i(t_i) g_i h_{i+1}(t_i)^{-1}$ for $i = 1, \ldots, k-1$, $g'_0 = g_0 h_1(0)^{-1}$ and $g'_k = h_k(1) g_k$.

The equivalence class of a $\mathcal{G}$-path c from x to y will be noted $[c]_{x,y}$, and the set of such equivalence classes will be noted $\Omega^c_{x,y}(\mathcal{G})$, or simply $\Omega^c_{x,y}$ (here c stands for continuous). It corresponds bijectively to the set of isomorphims classes of principal $\mathcal{G}$-bundles E over $I = [0,1]$ with two base points e_0 and e_1 over 0 and 1 such that $\alpha_E(e_0) = x$ and $\alpha_E(e_1) = y$ (see 2.2.3). The bundle E is obtained from c as the quotient of the union of the bundles $c^*_i(\mathcal{G})$ by the equivalence relation identifying $(t_i, g_i g) \in c^*_i(\mathcal{G})$ to $(t_i, g) \in c^*_{i+1}(\mathcal{G})$ for $i = 1, \ldots, k-1$. The base point e_0 is represented by $(0, g_0) \in c^*_1(\mathcal{G})$ and e_1 by $(1, g_k^{-1})$.

If $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ and $c' = (g'_0, c'_1, g'_1, \ldots, c'_k, g'_k)$ are two equivalent $\mathcal{G}$-paths from x to y over the same subdivision, then the maps h_i in 2.3.2, ii) above are unique, because $\mathcal{G}$ is Hausdorff and étale. Therefore there is a unique isomorphism from the relative principal $\mathcal{G}$-bundle associated to c to the one associated to c'.

Let $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ be a $\mathcal{G}$-path from x to y and g be an element of $\mathcal{G}$ with $\alpha(g) = x$ and $\omega(g) = z$ (resp. $\alpha(g) = z$ and $\omega(g) = y$). Then $g.c := (gg_0, c_1, g_1, \ldots, c_k, g_k)$ (resp. $c.g := (g_0, c_1, g_1, \ldots, c_k, g_k g)$) is a $\mathcal{G}$-path from z to y (resp. from x to z) whose equivalence class depends only on the equivalence class of c and which is noted $g.[c]_{x,y} \in \Omega^c_{z,y}$ (resp $[c]_{x,y}.g \in \Omega^c_{x,z}$).

2.3.3. Based $\mathcal{G}$-loops. The sets Ω^c_x and Ω^c_X

A continuous $\mathcal{G}$-path c from x to x is called a closed $\mathcal{G}$-path (based at x). Its

set of continuous $\mathcal{G}$-loops based at x is also noted $\Omega_x^c(\mathcal{G})$ or simply Ω_x^c. We note $\Omega_X^c(\mathcal{G})$ or simply Ω_X^c the set $\bigcup_{x\in X}\Omega_x^c$ of based continuous $\mathcal{G}$-loops.

The set Ω_x^c is in bijection with the set of isomorphisms classes of principal $\mathcal{G}$-bundle E over the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with a base point e over $0 \in \mathbb{S}^1$ such that $\alpha_E(e) = x$. A pointed principal $\mathcal{G}$-bundle (E, e) corresponding to c is constructed as follows. For $i = 1, \ldots, k$, let $E_i = [t_{i-1}, t_i] \times_X \mathcal{G}$ be the pull back of the principal $\mathcal{G}$-bundle $\mathcal{G}$ by the map $c_i : [t_{i-1}, t_i] \to X$, i.e. the space of pairs (t, g) with $c_i(t) = \omega(g)$; recall that the action map $\alpha_i : E_i \to X$ sends (t, g) to $\alpha(g)$. The bundle E is the quotient of the disjoint union of the E_i by the equivalence relation identifying $(t_i, g) \in E_{i+1}$ to $(t_i, g_i g) \in E_i$ for $1 \le i \le k$ and $(0, g) \in E_1$ to $(1, g_k g_0 g) \in E_k$. The base point e is the equivalence class of $(0, g_0^{-1}) \in E_1$, or equivalently of $(1, g_k) \in E_k$. The only automorphism of a pointed principal $\mathcal{G}$-bundle (E, e) over $\mathbb{S}^1$ (respecting the base point and projecting to the identity of $\mathbb{S}^1$) is the identity.

The groupoid $\mathcal{G}$ acts naturally on the left on the set Ω_X^c of based continuous $\mathcal{G}$-loops with respect to the projection $\Omega_X \to X$ associating to a $\mathcal{G}$-loop based at x the point x. Indeed let $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ be a closed $\mathcal{G}$-path based at x and g be an element of $\mathcal{G}$ with $\alpha(g) = x$ and $\omega(g) = y$. The action of g on the equivalence class of c is the $\mathcal{G}$-loop ${}^g[c]_x$ based at y represented by ${}^g c := (gg_0, c_1, g_1, \ldots, c_k, g_k g^{-1})$.

If $[c]_x$ is represented by the pointed principal $\mathcal{G}$-bundle (E, e), then ${}^g[c]_x$ is represented by the pointed bundle $(E, e.g^{-1})$.

For an integer m, the map $t \mapsto mt \mod 1$ from $\mathbb{S}^1$ to $\mathbb{S}^1$ induces a map $\Omega_x^c \to \Omega_x^c$ mapping $[c]_x$ to $[c]_x^m$. On the level of pointed bundles, it is just the pull back. Concretely when m is a positive integer, if $c = (g_0, c_1, \ldots, c_k, g_k)$ is a closed $\mathcal{G}$-path over a subdivision $0 = t_0 < \cdots < t_k = 1$, the image of $[c]_x$ by this map is equal to $[c^m]_x$, where c^m is the m-th iterate of c. More precisely, $c^m = (g'_0, c'_1, \ldots, g'_{mk})$ is the closed $\mathcal{G}$-path over the subdivision $0 = t'_0 < t'_1 < \cdots < t'_{mk} = 1$, where, for $r = 0, ..., m-1$ and $i = 1, \ldots, k$, we have $t'_{rk+i} = \frac{r}{m} + \frac{t_i}{m}$, $c'_{rk+i}(t) = c_i(mt - r)$; we have $g'_{(r+1)k} = g_k g_0$, $g'_0 = g_0$ and $g'_{mk} = g_k$, and for $i \neq k$ we have $g'_{rk+i} = g_i$.

2.3.4. The set $|\Lambda^c Q|$ of continuous free loops. It is the quotient $\mathcal{G}\backslash\Omega_X^c$ of Ω_X by the action of $\mathcal{G}$ described above. Its elements are called continuous free loops on Q. They correspond to the elements of $H^1(\mathbb{S}^1, \mathcal{G})$, the set of continuous morphisms from $\mathbb{S}^1$ to $\mathcal{G}$. The notation is justified by the observation that $H^1(\mathbb{S}^1, \mathcal{G}) = H^1(\mathbb{S}^1, \mathcal{G}')$ if $\mathcal{G}$ and $\mathcal{G}'$ are the groupoids of germs of changes of charts of two atlases of uniformizing charts defining the same orbifold structure on $|Q|$.

An element of $|\Lambda^c Q|$ is represented by a closed $\mathcal{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ over a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ as in 2.3.2 based at some point $x \in X$. This time the equivalence relation is generated by i) and ii) in 2.3.2 and also by

iii) for any element $g \in \mathcal{G}$ such that $\alpha(g) = x$, then $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ is equivalent to ${}^g c := (gg_0, c_1, g_1, \ldots, c_k, g_k g^{-1})$. The class of c under this equivalence relation is noted $[c]$

Under the projection $q : X \to |Q|$, every free $\mathcal{G}$-loop is mapped to a free loop on $|Q|$. Therefore if $\Lambda|Q|$ is the space of continuous free loops on the topological space $|Q|$ in the usual sense, we have a map

$$|\Lambda^c Q| \to \Lambda|Q|.$$

This map is not injective in general (see below).

We note $|\Lambda^0 Q|$ the subset of $|\Lambda^c Q|$ formed by the free loops on Q projecting

$c = (g_0, c_1, g_1)$, where g_0 is a unit 1_x, c_1 is the constant map from $[0,1]$ to x and g_1 is an element of the subgroup $\mathcal{G}_x = \{\{g \in \mathcal{G} : \alpha(g) = \omega(g) = x\}$. The equivalence class $[c]$ of c correspond to the conjugacy class of g_1 in $\mathcal{G}_x$.

Alternatively $|\Lambda^c Q|$ can be described as the set of isomorphism classes of principal $\mathcal{G}$-bundle E over $\mathbb{S}^1$. The group of homeomorphisms of $\mathbb{S}^1$ acts on $|\Lambda^c Q|$ by change of parametrisation: if h is a homeomorphism of $\mathbb{S}^1$, its action on the isomorphism class of a bundle E is the isomorphism class of the pull back of E by h. In particular the group $\mathbb{S}^1$ of rotations of $\mathbb{S}^1$ acts on $|\Lambda^c Q|$. The fixed point set of this action is precisely $|\Lambda^0 Q|$.

2.3.5. The developable case. Let Q be the orbifold quotient of a connected manifold X by the action of a discrete subgroup Γ of its group of diffeomorphisms acting properly and let $\mathcal{G}$ be the groupoid $\Gamma \ltimes X$ (see the end of 2.1.4). The set Ω^c_X of based continuous $\mathcal{G}$-loops are in bijection with the pairs (c, γ), where $c : [0,1] \to X$ is a continuous path and γ is an element of Γ mapping $c(0)$ to $c(1)$. Indeed, consider a $\mathcal{G}$-loop at x represented by the closed $\mathcal{G}$-path $(g_o, c_1, \ldots, c_k, g_k)$ over the subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$, where $g_i = (\gamma_i, c_{i+1}(t_i))$ for $i = 0, \ldots, k-1$ and $g_k = (\gamma_k, c_k(1))$. Then we define $\gamma = \gamma_0 \ldots \gamma_k$ and $c : [0,1] \to X$ as the path defined for $t \in [t_{i-1}, t_i]$ by $\gamma_0 \ldots \gamma_{i-1}.c_i(t)$. The continuous free loops on Q are represented by classes of pairs (c, γ) like above, such a pair being equivalent to $(\delta \circ c, \delta\gamma\delta^{-1})$, where $\delta \in \Gamma$. So $|\Lambda^c Q|$ is the quotient of Ω^c_X by this action of Γ. Assuming X simply connected, the set of homotopy classes of elements of $|\Lambda^c Q|$ is in bijection with the set of conjugacy classes in Γ.

We can equivalently describe the elements of Ω^c_X as the pairs (c, γ) where $c : \mathbb{R} \to X$ is a continuous map such that $c(t+1) = \gamma.c(t)$. With this interpretation, we can describe the action of $\mathbb{S}^1$ on $|\Lambda^c Q| = \Gamma \backslash \Omega^c_X$ as follows. We have a natural action of $\mathbb{R}$ on Ω^c_X by translations: the action of $\tau \in \mathbb{R}$ is given by $\tau.(c, \gamma) = (c_\tau, \gamma)$, where $c_\tau(t) = c(t + \tau)$. This action commutes with the action ${}^\delta(c, \gamma) := (\delta \circ c, \delta\gamma\delta^{-1})$ of $\delta \in \Gamma$ on Ω_X. Therefore we get an action of $\mathbb{R}$ on the quotient $\Gamma \backslash \Omega^c_X$. As the translations by the integers act trivially, we get the action of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ on $|\Lambda^c Q|$.

As an example, let $X = \mathbb{R}^2$ and Γ be the group generated by a rotation ρ fixing 0 and of angle $2\pi/n$. Let $\mathcal{G} = \Gamma \ltimes X$ be the groupoid associated to the action of Γ on X. The orbifold $Q = \mathcal{G} \backslash X$ is a cone. Consider the free $\mathcal{G}$-loop represented by the pair (c, ρ^k), where c is the constant path at 0. If we deform this loop slightly so that it avoids the origin, its projection to the cone $|Q|$ will be a curve going around the vertex a number of times congruent to k modulo n ; in particular, when $k = n$, it could also be a constant loop.

2.4 Geodesics on Riemannian orbifolds

2.4.1. Length and Energy. We consider a Riemannian orbifold $Q = \mathcal{G} \backslash X$. The length $L(c)$ of a $\mathcal{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ is the sum of the length of the paths c_i. It depends only on the equivalence class of c. If c is piecewise differentiable (i.e. if each c_i is piecewise differentiable), the length of c is given by

$$L(c) = \sum^{k} \int^{t_i} |\dot{c}_i(t)| \, dt,$$

and the energy $E(c)$ of c is

$$E(c) = 1/2 \sum_{1}^{k} \int_{t_{i-1}}^{t_i} |\dot{c_i}^2(t)| \, dt.$$

¿From Schwartz inequality we have

$$L(c)^2 \leq 2E(c),$$

with equality if and only if the speed $|\dot{c}_i(t)|$ is constant for all i and t. The length and the energy of c depends only on its equivalence class.

The distance $d(z, z')$ between two points $z = q(x)$ and $z' = q(x')$ in $|Q|$ is defined as the infimum of the length of $\mathcal{G}$-paths joining x to x' (this is independent of the choice of x and x' in their $\mathcal{G}$-orbit). This distance defines a metric on $|Q|$. The Riemannian orbifold Q is said to be complete, if this metric is complete. This is always the case if $|Q|$ is compact.

2.4.2. Geodesic $\mathcal{G}$-paths and closed geodesics on orbifolds.

A geodesic $\mathcal{G}$-path from x to y in a Riemannian orbifold is a $\mathcal{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ from x to y such that each c_i is a geodesic segment with constant speed and such that the differential Dg_{i+1} of a representative of g_{i+1} at $c_{i+1}(t_i)$, maps the velocity vector $\dot{c}_{i+1}(t_i)$ to the velocity vector $\dot{c}_i(t_i)$. Note that the image of a $\mathcal{G}$-geodesic path under the projection to $|Q|$ is an arc which in general is not locally length minimizing for the metric on $|Q|$ defined above (see the example in 2.4.5).

Note that if c is a geodesic $\mathcal{G}$-path from x to y, then the vector $Dg_0(\dot{c}_1(0))$ is an invariant of the equivalence class $[c]_{x,y}$ and is called the initial vector of the $\mathcal{G}$-geodesic c.

If c is a closed $\mathcal{G}$-path, it represents a closed $\mathcal{G}$-geodesic $[c]_x$ based at x if moreover the differential of $g_k g_0$ maps the velocity vector $\dot{c_1}(0)$ to the vector $\dot{c_k}(1)$. Its equivalence class $[c]$ is called a closed geodesic on Q. A free loop of length 0 is always a closed geodesic.

2.4.3. Geometric equivalence of closed geodesics. Two closed geodesic on Q are geometrically equivalent if their image under the projection to $|Q|$ are the same. Otherwise they are called geometrically distinct. The elements of the $\mathbb{S}^1$-orbit of a closed geodesic $[c]$ are all geometrically equivalent to $[c]$, as well as the multiples $[c^m]$ for $m \neq 0$ (see 2.3.3).

A closed geodesic is called primitive if its length is the minimum of the lengths of the closed geodesics in its geometric equivalence class.

2.4.4. The exponential morphism. If Q is a complete orbifold, given a vector $\xi \in T_x X$, there is always a geodesic $\mathcal{G}$-path $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ over a subdivision of $[0, 1]$ issuing from x with initial vector ξ. Any geodesic $\mathcal{G}$-path issuing from x with initial vector ξ is equivalent to a $\mathcal{G}$-path obtained from c by replacing g_k by any element $g_k g$ where $g \in \mathcal{G}$ is any element such that $\omega(g) = \alpha(g_k)$.

In fact we have the analogue of the *exponential map* for complete orbifolds, namely for each point $x \in X$ there is a morphism $exp_x : T_x X \to Q$ defined by a principal $\mathcal{G}$-bundle E_x over $T_x X$. The elements of E_x above $\xi \in T_x X$ are the equivalence classes of geodesic $\mathcal{G}$-paths with initial vector ξ. The right action of

$\mathcal{G}$ on E_x is as described above. There is a canonical base point e_x above $0 \in T_x$ characterized by the following property. Let s be the local section of E_x defined on a small ball $U \subset T_x X$ with center 0 mapping 0 to e; then its composition with α_{E_x} is the usual exponential map $U \to X$.

Given $g \in \mathcal{G}$ with $\alpha(g) = x$ and $\omega(g) = y$ there is a unique isomorphism $(E_x, e_x) \to (E_y, e_y)$ of pointed principal $\mathcal{G}$-bundle projecting to the differential $Dg : T_x X \to T_y X$ of g.

2.4.5. The developable case. If the orbifold Q is the quotient $\Gamma\backslash X$ of a connected Riemannian manifold X by a discrete subgroup Γ of its group of isometries, then any closed geodesic on Q is represented by a pair (c, γ), where $c : \mathbb{R} \to X$ is a geodesic and γ an element of Γ such that $\gamma.c(t) = c(t+1)$ for all $t \in \mathbb{R}$; in the terminology of K. Grove [6], c is called a γ-invariant geodesic. Another such pair (c', γ') represents the same closed geodesic on Q if and only if there is an element $\overline{\gamma} \in \Gamma$ such that $c' = \overline{\gamma}.c$ and $\gamma' = \overline{\gamma}\gamma\,\overline{\gamma}^{-1}$.

As an example, consider the orbifold Q which is the quotient of the round 2-sphere $\mathbb{S}^2$ by a rotation ρ of angle π fixing the north pole N and the south pole S. The quotient space $|Q|$ looks like a rugby ball, with two conical points $[N]$ and $[S]$, images of N and S. There are two homotopy classes of free loops on Q. Closed geodesics homotopic to a constant loop are represented by a closed geodesic on S^2 (their length is an integral multiple of 2π). If they have positive length, their image in $|Q|$ is either the equator, a figure eight or a meridian (image of a great circle through N and S). Closed geodesics in the other homotopy class are represented by a pair (c, ρ), where c is either the constant map to N or S, or maps $[0,1]$ to a geodesic arc on the equator of length an integral odd multiple of π.

2.4.6. Proposition. *Let Q be a compact orbifold. There is an $a > 0$ such that any closed geodesic of energy $< a$ has zero length.*

Proof. Using the compactness of $|Q|$, one can find a finite number of convex geodesic balls $X_1, \ldots, X_r$ in X such that the pseudogroup of change of charts restricted to X_i is generated by a finite group Γ_i of isometries of X_i, and such that the union of the $q(X_i)$ is $|Q|$. In particular there are no closed geodesics of positive length contained in X_i. Using again the compactness of $|Q|$ one can find a positive number ρ such that, for every point $z \in |Q|$, there is a point x in some X_i with $z = q(x)$ which is the center of a geodesic ball of radius ρ contained in X_i.

Choose $a < \rho^2/2$. Any closed geodesic with energy $< a$ is represented by a pair (c, γ), where $c : [0,1] \to X$ is a geodesic segment of length $< \rho$ contained in some X_i and $\gamma \in \Gamma_i$ with $\dot{c}(1) = \gamma.\dot{c}(0)$. If m is the order of γ, then the m-th iterate c^m of this closed $\mathcal{G}$-geodesic as defined in 2.3.3, gives a closed geodesic contained in X_i, hence is of length zero. Therefore c is a comstant map. $\square$

2.5 Classifying spaces

For any topological groupoid $(\mathcal{G}, X)$, one can construct a classifying space $B\mathcal{G}$, base space of a principal $\mathcal{G}$-bundle $E\mathcal{G} \to B\mathcal{G}$. One possible construction is the geometric realization of the nerve of the topological category $\mathcal{G}$. This construction is functorial with respect to continuous homomorphisms of groupoids. When $\mathcal{G}$ is the groupoid of germs of changes of charts of an atlas of uniformizing charts for a Riemannian orbifold Q of dimension n, there is an explicit construction of $B\mathcal{G}$ which is independent of the particular atlas defining Q and which will be therefore noted BQ (see [9].)

2.5.1. Construction of the classifying space BQ. Consider the bundle of orthonormal coframes FX on X; an element of FX above $x \in X$ can be identified to a linear isometry from the tangent space T_xX at x to the Euclidean space $\mathbb{R}^n$; the group $O(n)$ of isometries of $\mathbb{R}^n$ acts naturally on the left on FX and this action commutes with the right action of the groupoid $\mathcal{G}$ on FX through the composition with the differential of the elements of $\mathcal{G}$. As the action of $\mathcal{G}$ on FX is free, the quotient $FX/\mathcal{G}$ is a smooth manifold FQ depending only on Q and not of a particular atlas defining Q. The left action of $O(n)$ on FX gives a locally free action of $O(n)$ on $FX/\mathcal{G} = FQ$.

Choose a principal universal $O(n)$-bundle $EO(n) \to BO(n)$ for the orthogonal group $O(n)$ and take for $E\mathcal{G}$ the associated bundle $EO(n) \times_{O(n)} FX$, quotient of $EO(n) \times FX$ by the diagonal action of $O(n)$. The projection from FX to X gives a projection $\alpha_{E\mathcal{G}} : E\mathcal{G} \to X$ with contractible fibers isomorphic to $EO(n)$. As the action of $O(n)$ on FX commutes with the natural right action of $\mathcal{G}$, we get a free action of $\mathcal{G}$ on $E\mathcal{G}$ with respect to the projection $\alpha_{E\mathcal{G}}$, and $E\mathcal{G} \to E\mathcal{G}/\mathcal{G} = EO(n) \times_{O(n)} FX/\mathcal{G}$ is a universal principal $\mathcal{G}$-bundle whose base space $B\mathcal{G} = E\mathcal{G}/\mathcal{G} = EO(n) \times_{O(n)} FQ$ will be noted BQ. There is a canonical map $\pi : BQ \to |Q|$ induced from the map $q \circ \alpha_{E\mathcal{G}} : E\mathcal{G} \to X/\mathcal{G} = |Q|$; it is the projection of the morphism from BQ to $\mathcal{G}$ associated to the principal $\mathcal{G}$-bundle $E\mathcal{G} \to BQ$; the fiber of π above a point $z = q(x)$ is an acyclic space with fundamental group isomorphic to the isotropy subgroup $\mathcal{G}_x$ of x.

2.5.2. What does it classify? The "maps" from a polyhedron K to Q (i.e. the morphisms from K to $\mathcal{G}$. see 2.2.1, or equivalently the isomorphisms classes of principal $\mathcal{G}$-bundle over K, see 2.2.2) correspond bijectively (see [9]) to the equivalence classes of continuous maps from K to BQ, two such maps being equivalent if they are connected by an homotopy along the fibers of the projection $BQ \to |Q|$. If L is a subpolyhedron of K and F a principal $\mathcal{G}$-bundle over L, isomorphic to the pull back of $E\mathcal{G} \to B\mathcal{G}$ by a continuous maps $f_L : L \to B\mathcal{G}$, the morphisms from K to $\mathcal{G}$ relative to F correspond bijectively to the equivalence classes of continuous maps $f : K \to B\mathcal{G}$ whose restriction to L is f_L, two such maps being equivalent if they are connected by an homotopy along the fibers of the projection to $X/\mathcal{G} = |Q|$ which is fixed on L (cf [9]). This property, called universal property, characterizes a classifying space for $\mathcal{G}$ up to weak homotopy equivalence.

2.5.3. Homology properties. The projection $BQ \to |Q|$ induces an isomorphism on rational homology (or cohomology), because the fibers have trivial rational homology. If Q is a connected compact orientable orbifold of dimension n, then an orientation determines a fundamental integral class which is a generator of $H_n(|Q|, \mathbb{Z}) = \mathbb{Z}$. By the isomorphism $H_n(BQ, \mathbb{Q}) \cong H_n(|Q|, \mathbb{Q})$, this class corresponds to a generator of $H_n(BQ, \mathbb{Q}) = \mathbb{Q}$ called the fundamental class of the oriented orbifold Q.

The projection $BQ \to |Q|$ induces a surjective homomorphism on the fundamental groups. In general the homotopy groups of BQ, which are isomorphic to the homotopy groups of Q, in the orbifold sense (see 2.2.3 and 2.5.2), are quite different from the homotopy groups of $|Q|$.

3. The free loop space of a Riemannian orbifold

quotient $\mathcal{G}\backslash X$ (X can be the disjoint union of the sources of an atlas of uniformizing charts and $\mathcal{G}$ is the groupoid of germs of change of charts). We recall that $\Omega^c_X = \bigcup_{x\in X}\Omega^c_x$ is the union of the sets Ω^c_x of continuous $\mathcal{G}$-loops based at x. The groupoid $\mathcal{G}$ acts on Ω^c_X and the quotient by this action is the set $|\Lambda^c Q|$ of continuous free loops on Q.

3.1 The Banach orbifold $\Lambda^c Q$

3.1.1. Proposition. *The set Ω^c_X of continuus based $\mathcal{G}$-loops, as well as the set $\Omega^c_{x,y}$ of equivalence classes of continuous $\mathcal{G}$-paths from x to y, has a natural structure of Banach manifold.*

Proof. On Ω^c_X the structure of Banach manifold is constructed as follows. Let $c = (g_0, c_1, g_1, \ldots, c_k, g_k)$ be a closed $\mathcal{G}$-path over the subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ based at x (the target of g_0). Let c^*TX be the vector bundle over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ which is the quotient of the disjoint union of the bundles c_i^*TX by the equivalence relation which identifies the point $(t_i, \xi_i) \in c_i^*TX$ with the point $(t_i, Dg_i(\xi_i)) \in c^*_{i-1}TX$ for $1 < i \leq k$ and $(0, \xi_0) \in c_1^*TX$ with $(1, D(g_k g_0)(\xi_0)) \in c_k^*TX$. The projection to the base space $\mathbb{S}^1$ maps the equivalence class of (t, ξ) to t modulo 1. On the fibers we have a scalar product induced from the scalar product given on the fibers of TX by the Riemannian structure on X. If c' is a closed $\mathcal{G}$-path based at x equivalent to c, there is a natural isomorphism between c^*TX and c'^*TX. The tangent space of Ω^c_X at $[c]_x$ will be the Banach space $C^0(\mathbb{S}^1, c^*TX)$ of continuous sections of the bundle c^*TX with the sup norm. Such a section v is represented by a sequence $(v_1, \ldots, v_k)$, where v_i is a vector field along c_i such that the above compatibility conditions are satisfied at the points t_i.

Given c, choose $\epsilon > 0$ so small that, for each $t \in [t_{i-1}, t_i]$, the exponential map $\exp_{c_i(t)}$ is defined and is injective on the ball of radius ϵ in $T_{c_i(t)}X$; then g_i extends uniquely (see 2.1.5) to a section $\tilde{g}_i$ of α defined on the ball of radius ϵ and center $\alpha(g_i)$. Let $\tilde{U}^\epsilon_c$ be the open ball of radius ϵ in $C^0(\mathbb{S}^1, c^*TX)$ centered at the origin. We define the chart $\exp^\epsilon_c : \tilde{U}^\epsilon_c \to U^\epsilon_c \subseteq \Omega_X$ by mapping a section v given by $(v_1, \ldots, v_k)$ to the equivalence class of the based closed $\mathcal{G}$-path $c^v = (g^v_0, c^v_1, g^v_1, \ldots, c^v_k, g^v_k)$ defined as follows: $c^v_i(t) = \exp_{c_i(t)} v_i(t)$, $g^v_i = \tilde{g}_i(c^v_{i+1}(t_i))$ for $i < k$ and $g^v_k = \tilde{g}_k(\omega(g^v_0))$.

It is easy to see that each $\exp^\epsilon_c$ is a bijection (see 2.3.2), that the images U^ϵ_c for various c and ϵ form a basis for a topology on Ω^c_X, and that the change of charts are differentiable.

Therefore Ω^c_X is a Banach manifold. The tangent space $T_{[c]_x}\Omega^c_X$ at $[c]_x$ is the space of continuous sections of the vector bundle c^*TX over $\mathbb{S}^1$. Note that Ω^0_X is a finite dimensional submanifold of Ω^c_X.

The Banach manifold structure on $\Omega^c_{x,y}$ is defined similarly. For a $\mathcal{G}$-path $c = (g_0, c_1, \ldots, c_k, g_k)$, the tangent space at $[c]_{x,y}$ is isomorphic to the space of vector fields $v = (v_1, \ldots, v_k)$ along c which vanish at 0 and 1. □

The action of $\mathcal{G}$ on Ω^c_X with respect to the projection assigning to a based $\mathcal{G}$-loop its base point is differentiable. The quotient of Ω^c_X by this action is by definition the "space" of (continuous) free loops $|\Lambda^c(\mathcal{G})| = |\Lambda^c Q|$ on Q.

3.1.2. Proposition. *The space $|\Lambda^c Q|$ of continuous free $\mathcal{G}$-loops on Q has a natural Banach orbifold structure noted $\Lambda^c Q$. The subspace $|\Lambda^0 Q|$ of free loops of length zero is a "suborbifold" $\Lambda^0 Q$ of $\Lambda^c Q$.*

Proof. If $q_i : X_i \to V_i$ is a uniformizing chart for Q (see 2.1.1), then $\Omega^c_{X_i} \to \Gamma_i \backslash \Omega^c_{X_i}$ is a uniformizing chart for $\Lambda^c Q$, where $\Omega^c_{X_i} = \bigcup_{x \in X_i} \Omega^c_x$. The groupoid of germs of changes of chart is the groupoid $\overline{\mathcal{G}} := \mathcal{G} \times_X \Omega^c_X$, the subspace of $\mathcal{G} \times \Omega^c_X$ consisting of pairs $(g, [c]_x)$ with $\alpha(g) = x$. The source (resp. target) projection maps $(g, [c]_x)$ to $[c]_x$ (resp. ${}^g[c]_x$). The composition $(g', [c']_{x'})(g, [c]_x)$ is defined if ${}^g[c]_x = [c']_{x'}$ and is equal to $(g'g, [c]_x)$.

The suborbifold structure on $\Lambda^0 Q$ is obtained by replacing Ω^c_X by Ω^0_X and $\mathcal{G} \times_X \Omega^c_X$ by $\mathcal{G} \times_X \Omega^0_X$. Note that $\Lambda^0 Q$ was considered by Kawasaki in [10]. $\square$

3.1.3. The action of $\mathbb{S}^1$ on the orbifold $\Lambda^c Q$. The action of $\mathbb{S}^1$ on $|\Lambda^c Q|$ described at the end of 2.3.4 comes from a continuous action of $\mathbb{S}^1$ on the orbifold $\Lambda^c Q$. This means that if c is a continuous $\mathcal{G}$-loop based at x and if c' is a $\mathcal{G}$-loop based at x' representing the translate $\tau.[c]$ of $[c]$ by $\tau \in \mathbb{R}\backslash\mathbb{Z} = \mathbb{S}^1$, there is a diffeomorphism h_τ of a neighbourhood of $[c]_x$ to a neighbourhood of $[c]_{x'}$ projecting to the translation by τ of a neighbourhood of $[c]$ to a neighbourhood of $[c']$ depending continuously on τ. As explained for instance in [12, p. 39], in the charts $\exp^\epsilon_c$ and $\exp^\epsilon_{c'}$, indeed h_τ is given by a linear map. The continuity in τ is checked as in [12] using a chart associated to a C^∞ based $\mathcal{G}$-loop close to $[c]_x$.

3.2 A classifying space for $\Lambda^c Q$

We consider as in 2.5 a classifying space BQ, base space of a universal principal $\mathcal{G}$-bundle $E\mathcal{G} \to B\mathcal{G} = BQ$. Let $E\mathcal{G} \times_X \Omega^c_X$ be the subspace of $E\mathcal{G} \times \Omega^c_X$ consisting of pairs $(e, [c]_x)$ such that $\alpha_{E\mathcal{G}}(e) = x$. We note $E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X$ its quotient by the equivalence relation identifying $(e.g, [c]_x)$ to $(e, {}^g[c]_x)$. We can consider $E\mathcal{G} \times_X \Omega^c_X \to E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X$ as a principal $(\mathcal{G} \times_X \Omega^c_X)$-bundle, the action map being the natural projection to Ω^c_X.

3.2.1. Proposition. *$E\mathcal{G} \times_X \Omega^c_X \to E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X$ is a principal universal $(\mathcal{G} \times_X \Omega^c_X)$-bundle. The base space $E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X$ will be noted $B\Lambda^c Q$.*

Similarly $E\mathcal{G} \times_X \Omega^0_X \to E\mathcal{G} \times_{\mathcal{G}} \Omega^0_X$ is a principal universal $(\mathcal{G} \times_X \Omega^0_X)$-bundle. Its base space $E\mathcal{G} \times_{\mathcal{G}} \Omega^0_X$ is noted $B\Lambda^0 Q$.

The natural projection $B\Lambda^c Q \to BQ$ (i.e. $E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X \to E\mathcal{G}/\mathcal{G} = B\mathcal{G} = BQ$) is a Serre fibration with fibers isomorphic to Ω^c_x.

Proof. With respect to the action map to X, $E\mathcal{G}$ is a locally trivial bundle with contractible fibers. The pull back of this bundle by the projection $\Omega^c_X \to X$ is the bundle $E\mathcal{G} \times_X \Omega^c_X$ with base space Ω^c_X. It has also contractible fibers and therefore $E\mathcal{G} \times_X \Omega^c_X \to E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X$ is a universal principal $(\mathcal{G} \times_X \Omega^c_X)$-bundle. The same argument works with Ω^0_X replacing Ω^c_X. It is easy to see that the projection $\Omega^c_X \to X$ is a Serre fibration. Therefore the projection $E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X \to B\mathcal{G}$ is also a Serre fibration because it is locally a pull back of the fibration $\Omega^c_X \to X$, as it is seen using a local section of the projection $E\mathcal{G} \to B\mathcal{G}$. $\square$

Let ΛBQ be the space of continuous free loops on BQ.

3.2.2. Theorem. *ΛBQ is the base space of a universal $(\mathcal{G} \times_X \Omega^c_X)$-bundle. This bundle is the pull back of $E\mathcal{G} \times_X \Omega^c_X$ by a map $\phi : \Lambda BQ \to B\Lambda^c Q$ which is a weak homotopy equivalence and commutes with the projections to BQ. Therefore ΛBQ is a classifying space for the orbifold $\Lambda^c Q$.*

For points $z \in BQ$ and $x \in X$ projecting to the same point of $|Q|$, the map ϕ induces a weak homotopy equivalence from the space $\Omega_z BQ$ of loops on BQ based at z to Ω^c_x.

We first prove two lemmas. The first one is a tautology.

3.2.3. Lemma. *There is a canonical principal $\mathcal{G}$-bundle $E_{\Omega^c_X}$ over $\Omega^c_X \times \mathbb{S}^1$ relative to the bundle over $\Omega^c_X \times \{1\}$ pull back of the bundle $\mathcal{G} \to X$ by the projection $([c]_x, 1) \mapsto x$ to X. Its restriction to $\{[c]_x\} \times \mathbb{S}^1$ is the relative principal $\mathcal{G}$-bundle $E_{[c]_x}$ over $\mathbb{S}^1$ associated to $[c]_x$ (cf. 2.3.2).*

Proof. Locally the bundle is constructed above the product of a neighbourhood U of $[c]_x$ with $\mathbb{S}^1$ as in 2.3.2 using a chart, and those local constructions are uniquely glued together uisng 2.3.3. □

3.2.4. Lemma. *Let K be a topological space and L be a subspace of K. Let $f_0 : L \to \Omega_x$ be a continuous map and let F be the principal $\mathcal{G}$-bundle over $A := (K \times \{1\}) \cup (L \times \mathbb{S}^1) \subseteq K \times \mathbb{S}^1$ whose restriction to $L \times \mathbb{S}^1$ is the pull back of $E_{\Omega^c_X}$ by the map $f_0 \times id$ and whose restriction to $K \times \{1\}$ is the pull back from $\mathcal{G}$ by the constant map to x.*

There is a bijective correspondence between continuous maps $f : K \to \Omega^c_x$ extending f_0 and isomorphism classes of principal $\mathcal{G}$-bundle over $K \times \mathbb{S}^1$ relative to F.

Proof. Given f, it is clear that the pull back of $E_{\Omega^c_x}$ by $f \times id$ is a bundle relative to F_0. Conversely, given such a bundle E, for each $y \in K$, its restriction to $\{y\} \times \mathbb{S}^1$ gives a $\mathcal{G}$-loop based at x that we note $f(y)$. It is clear that the map $f : K \to \Omega^c_x$ extends f_0. □

Proof of the theorem. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $B\mathcal{G} = BQ$ with local sections $s_i : U_i \to E\mathcal{G}$ with respect to the projection $E\mathcal{G} \to B\mathcal{G}$. Let $f_{ij} : U_i \cap U_j \to \mathcal{G}$ be the corresponding cocycle: for $z \in U_i \cap U_j$, we have $s_j(z) = s_i(z) f_{ij}(z)$; in particular, $f_i := f_{ii} : U_i \to X$ is equal to $\alpha_{E\mathcal{G}} \circ s_i$.

Let $\pi : \Lambda B\mathcal{G} \to B\mathcal{G}$ be the map associating to a loop $l : [0,1] \to B\mathcal{G}$ its base point $l(0) = l(1)$. For each $i \in I$, we consider the continuous map $\phi_i : \pi^{-1}(U_i) \to \Omega_X$ defined as follows. Given a loop l on BQ based at $z \in U_i$, we choose a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that $l([t_{j-1}, t_j]) \subseteq U_{r_j}$ for some $r_j \in I$. Then $\phi_i(l)$ is the $\mathcal{G}$-loop based at $f_i(z)$ represented by the $\mathcal{G}$-path $c = (g_0, c_1, \ldots, c_k, g_k)$ where $c_j : [t_{j-1}, t_j] \to X$ maps t to $f_{r_j}(l(t))$, $g_j = f_{r_j r_{j+1}}(l(t_j))$ for $1 \le j \le k-1$, $g_0 = f_{ir_1}(l(0))$ and $g_k = f_{r_k i}(l(1))$. It is easy to see that the equivalence class of c does not depend of the choices and that the map ϕ_i is continuous.

By construction the restriction of the map ϕ to $\pi^{-1}(U_i)$ is defined as the composition of $(s_i, \phi_i) : \pi^{-1}(U_i) \to E\mathcal{G} \times_X \Omega^c_X$ with the projection to the quotient $E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X = B\Lambda^c Q$.

The spaces ΛBQ and $B\Lambda^c Q$ are Serre fibrations over BQ and ϕ commutes with the projections. To prove that ϕ is a weak homotopy equivalence it is sufficient to check that, for a base point $z \in U_i$, the restriction of ϕ to the space $\pi^{-1}(z) = \Omega_z(BQ)$ of loops on BQ based at z is a weak homotopy equivalence to the fiber of $E\mathcal{G} \times_{\mathcal{G}} \Omega^c_X \to BQ$ above z. This fiber is isomorphic to Ω^c_x, the space of continuous $\mathcal{G}$-loops based at $x = f_i(z)$, by the map sending $[c]_x$ to the class modulo $\mathcal{G}$ of $(s_i(z), [c]_x)$. Via this isomorphism, ϕ maps a loop $l \in \Omega^c_z$ to $\phi_i(l)$.

Fix a base point $l \in \Omega_z BQ$ and let $[c]_x$ be its image by ϕ_i. We shall apply below the lemma 3.2.4 with $K = \mathbb{S}^m$, $L = *$ a base point and f_0 mapping $*$ to $[c]_x$. We first observe that the set $\pi_m(\Omega_z BQ; l)$ is in bijection with the set of homotopy

classes of maps from $\mathbb{S}^m \times \mathbb{S}^1$ to BQ which maps $\mathbb{S}^m \times \{0\}$ to z and whose restriction to $\{*\} \times \mathbb{S}^1$ is equal to l. According to 2.5.2, this set is in bijection with the set of homotopy classes of principal $\mathcal{G}$-bundles over $\mathbb{S}^m \times \mathbb{S}^1$ relative to F. Using 3.2.4, we see that this set corresponds bijectively to the set of homotopy classes of maps $(\mathbb{S}^m, *) \to (\Omega^c_x, [c]_x)$. $\square$

3.2.5. Remark. Composing $\phi : \Lambda BQ \to B\Lambda^c Q$ with the natural projection from $B\Lambda^c Q$ to $|\Lambda^c Q|$, we get an $\mathbb{S}^1$-equivariant map

$$\Lambda BQ \to |\Lambda^c Q|$$

with respect to the natural action of $\mathbb{S}^1$ on free loops.

3.2.6. Remark. One can show that, for a space K, there is a canonical correspondance associating to a principal $\mathcal{G}$-bundle E over $K \times \mathbb{S}^1$ a principal $\overline{\mathcal{G}} -$ *bundle* over K, where $\overline{\mathcal{G}} = \mathcal{G} \times_X \Omega^c_X$, inducing a bijection on isomorphisms classes. The universal $\overline{\mathcal{G}}$-bundle over ΛBQ corresponds to the principal $\mathcal{G}$-bundle over $\Lambda BQ \times \mathbb{S}^1$ which is the pull back of $E\mathcal{G}$ by the evaluation map $\Lambda BQ \times \mathbb{S}^1 \to BQ$ sending (l, t) to $l(t)$.

3.3 The Riemannian orbifold ΛQ of free $\mathcal{G}$-loops of class H^1

We consider as above a Riemannian orbifold $Q = \mathcal{G} \backslash X$.

3.3.1. $\mathcal{G}$-paths of class H^1. A $\mathcal{G}$-path $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ over a subdivision $0 = t_0 < t_1 < \dots < t_k = 1$ is of class H^1 if each c_i is absolutely continuous and the velocity functions $t \mapsto |\dot{c}_i(t)|$ are square integrable. Those conditions are also satisfied by any $\mathcal{G}$-path in the equivalence class of c. We denote $\Omega_{x,y}$ the set of equivalence classes of $\mathcal{G}$-paths of class H^1 from x to y and $|\Lambda Q|$ (resp. Ω_x) the set of free $\mathcal{G}$-loops on Q (resp. based at x) represented by closed $\mathcal{G}$-path of class H^1. Let $\Omega_X = \bigcup_{x \in X} \Omega_x$. The energy function E is defined on all those spaces.

3.3.2. $\Omega_{x,y}$ and Ω_X as Riemannian Hilbert manifolds.
On Ω_X, we define a structure of Hilbert Riemannian manifold using the following charts. Let $c = (g_0, c_1, g_1, \dots, c_k, g_k)$ be a closed $\mathcal{G}$-path over the subdivision $0 = t_0 < t_1 < \dots < t_k = 1$; we assume that c is piecewise differentiable, i.e. that each c_i is differentiable. We consider as in 3.1.1 the vector bundle c^*TX over $\mathbb{S}^1$ which is the union of the vector bundles c_i^*TX on which we consider the connection induced from the Levi Civita connection on TX. A continuous section $v = (v_1, \dots, v_k)$ of c^*TM is an H^1-section if each v_i is absolutely continuous and the covariant derivative ∇v_i is square integrable. The space of H^1-sections is a separable Hilbert space noted $H^1(c^*TX)$: if $w = (w_1, \dots, w_k)$ is another H^1-section, then the scalar product is defined by the following formula

$$(v, w) = \sum_i \int_{t_{i-1}}^{t_i} [\langle v_i(t), w_i(t) \rangle + \langle \nabla v_i(t), \nabla w_i(t) \rangle] dt. \tag{3.3.3}$$

This definition is independent of the particular choice of c in its equivalence class.

Consider the open set of $H^1(c^*TM)$ consisting of sections v such that $|v_i(t)| < \epsilon$, where ϵ is like in 3.1.1. The map $\exp_c$ maps this open set to a subset ${U'}^{\epsilon}_c$ of Ω_X. When c varies over the closed piecewise differentiable $\mathcal{G}$-paths, one proves like in the classical case of Riemannian manifolds (see for instance [11,12,]) that one obtains in

this way an atlas defining on Ω_X a structure of Hilbert manifold. When c is a closed $\mathcal{G}$-path based at x which is of class H^1, then the bundle c^*TX has also a natural connection induced from the connection on TX, and one can also define, using the formula 3.3.3, the Hilbert space $H^1(c^*TX)$ of H^1-sections of this vector bundle (namely sections $v = (v_1, \dots, v_k)$ such that each v_i gives a map from $[t_{i-1}, t_i]$ to TX which is of class H^1); it is canonically isometric to the tangent space of Ω_X at the equivalence class $[c]_x$ of c.

Similarly the set $\Omega_{x,y}$ is naturally a Riemannian manifold, the tangent space at $[c]_{x,y}$ is the space of H^1-sections of the bundle $c^*(TX)$ over $[0,1]$ which vanish at 0 and 1, with the scalar product given by 3.3.3. In particular Ω_x is a closed submanifold of Ω_X. It is complete as a Riemannian manifold if the orbifold Q is complete.

3.3.4. The Riemannian orbifold ΛQ. Like in 3.1.2 the space $|\Lambda Q| = \mathcal{G}\backslash\Omega_X$ of free $\mathcal{G}$-loops on Q of class H^1 as a natural Riemannian orbifold structure noted ΛQ. The groupoid of germs of change of charts for ΛQ is $\mathcal{G} \times_X \Omega_X$. As in 3.2.1 one proves that the space $B\Lambda Q := E\mathcal{G} \times_{\mathcal{G}} \Omega_X$ is a classifying space for ΛQ.

Like in 3.1.3 one can check (see [12, p.39] that the action of the group $\mathbb{S}^1$ on $|\Lambda Q|$ comes from an action of $\mathbb{S}^1$ on the orbifold ΛQ by isometries which is continuous (but not differentiable).

3.3.5 Proposition. *The natural inclusions*

$$\Omega_X \to \Omega_X^c, \qquad \Omega_{x,y} \to \Omega_{x,y}^c$$

are continuous and are homotopy equivalences. In particular, if Q is connected, $\Omega_{x,y}$ has the same weak homotopy type as the space of loops on BQ based at a fixed point.

The induced inclusion

$$B\Lambda Q \to B\Lambda^c Q$$

is a homotopy equivalence.

Proof. Denote by P the spaces Ω_X or $\Omega_{x,y}$ and by P^c the spaces Ω_X^c or $\Omega_{x,y}^c$. The fact that the inclusions $i : P \to P^c$ are continuous is proved like in the classical case. To show that they are homotopy equivalences, we follow the argument of Milnor [13, p. 93-94]. For a positive integer k, let P_k (resp.P_k^c) be the subspace of P (resp. P^c) formed by the elements represented by $\mathcal{G}$-paths $c = (g_0, c_1, g_1, \dots, c_{2^k}, g_{2^k})$ defined over the subdivision $0 = t_0 < \dots < t_{2^k} = 1$, where $t_i = i/2^k$, and such each $c_i(t_i)$ is the center of a convex geodesic ball containing the image of c_i. One can deform continuously (see [13, p. 91]) such a $\mathcal{G}$-path c to the $\mathcal{G}$-path $\overline{c} = (g_0, \overline{c}_1, g_1, \dots, \overline{c}_{2^k}, g_{2^k})$, where $\overline{c}_i$ is the geodesic segment joining $c_i(t_{i-1})$ to $c_i(t_i)$. Passing to equivalence classes, this gives a continuous deformation of $i_{|P_k} : P_k \to P_k^c$ and implies that this inclusion is a homotopy equivalence. As the spaces P and P^c are the increasing union of the open subspaces P_k and P_k^c for $k = 1, 2, \dots$, it follows that i is a homotopy equivalence (see the appendix of [13]).

The last assertion follows from the fact that the above deformation commutes with the projection to X and with the action of $\mathcal{G}$. □

3.3.6. The developable case. Let X be a simply connected Riemannian manifold, Γ a discrete subgroup of the group of isometries of X, and let Q be the

quotient orbifold $\Gamma\backslash X$. For $\gamma \in \Gamma$, the space of H^1-curves $c : [0,1] \to X$ such that $c(1) = \gamma.c(0)$ is noted $\Lambda(X, \gamma)$ in Grove-Tanaka [6]. It is connected because X is assumed to be simply connected. The space $\Omega_X = \Omega_X(\Gamma \ltimes X)$ of based $\Gamma \ltimes X$-loops of class H^1 is the disjoint union

$$\Omega_X = \coprod_{\gamma \in \Gamma} \Lambda(X, \gamma).$$

The connected components of the space $|\Lambda Q|$ of free loops of class H^1 are in bijection with the conjugacy classes in Γ. To the conjugacy class of $\gamma \in \Gamma$ corresponds the quotient of $\Lambda(X, \gamma)$ by the action of the centralizer Z_γ of γ in Γ.

Grove and Tanaka consider the case where X is compact simply connected. In that case Γ is a finite group. For $\gamma \in \Gamma$, the existence of one (resp. infinitely many geometrically distinct) γ-invariant geodesic of positive length is equivalent to the existence of one (resp. infinitely many geometrically distinct) closed geodesic on Q of positive length in the connected component $Z_\gamma\backslash\Lambda(X, \gamma)$ of ΛQ.

3.3.7 Classification of tubular neighbourhoods of $\mathbb{S}^1$-orbits. We consider an effective continuous action of $\mathbb{S}^1$ by isometries on a Riemannian orbifold. We have in mind the case of the natural action by isometries of $\mathbb{S}^1$ on ΛQ. We want to describe an invariant tubular neighbourhood of a smooth $\mathbb{S}^1$-orbit of positive length. We can assume that the orbifold is developable because such a tube is always developable. So we consider an orbifold Q which is the quotient of a connected Riemannian manifold Y by a discrete group Γ of isometries of Y acting properly. We assume that we have a continuous effective action of $\mathbb{S}^1$ on Q by isometries. This is equivalent to say that we have a continuous action of $\mathbb{R}$ on Y by isometries which commutes with the action of Γ and such that there is a unique element $\gamma_0 \in \Gamma$ such that, for each $y \in Y$, the translate $T_1 y$ of y by $1 \in \mathbb{R}$ is equal to $\gamma_0.y$. This implies that γ_0 is in the center of Γ. We note below $T_\tau y$ the translate of y by $\tau \in \mathbb{R}$.

Let y_0 be a point of Y which is not fixed by the action of $\mathbb{R}$, and let γ_0 be as above. We assume that the $\mathbb{R}$-orbit of y_0 is differentiable. Let $\overline{\Gamma}$ be the subgroup of Γ leaving invariant the $\mathbb{R}$-orbit of y_0. Consider the subgroup G of $\overline{\Gamma} \times \mathbb{R}$ consisting of pairs (γ, τ) such that $\gamma.y_0 = T_\tau y_0$. The image of G by the natural projection $(\gamma, \tau) \mapsto \tau$ is a discrete subgroup of $\mathbb{R}$ generated by a number $1/r$, where r is a positive integer. The kernel of this projection is canonically isomorphic to the finite subgroup Γ_0 of $\overline{\Gamma}$ fixing y_0. We have an exact sequence

$$1 \to \Gamma_0 \to G \xrightarrow{\psi} \mathbb{Z}\ 1/r \to 0.$$

Let B be the image by the exponential map $\exp_{y_0}$ of a small ball centered at 0 in the subspace of the tangent space at y_0 orthogonal to the velocity vector of the $\mathbb{R}$-orbit of y_0. In particular B is a slice for the $\mathbb{R}$-action and is left invariant by Γ_0.

3.3.8. Proposition. *Let I be the quotient of G by the subgroup generated by $(\gamma_0, 1)$. The group I acts effectively on B and the homomorphism ψ gives (modulo 1) a homomorphism from I to the subgroup of order r of $\mathbb{S}^1$. A $\mathbb{S}^1$-invariant tubular neighbourhood U of the $\mathbb{S}^1$-orbit of $[y_0]$ in $|Q|$ is $\mathbb{S}^1$-equivariantly isomorphic as an orbifold to the quotient $\mathbb{S}^1 \times_I B$ of $\mathbb{S}^1 \times B$ by the diagonal action of I.*

Proof. The group G acts on B (resp. $\mathbb{R}$), the element $(\gamma, \tau) \in G$ mapping $b \in B$ to $\gamma T_\tau^{-1}(b)$ (resp. t to $t + \tau$). Note that $(\gamma_0, 1)$ acts trivially on B. Consider

the immersion $i : \mathbb{R} \times B \to Y$ mapping (t, b) to $T_t b$. Its image is an $\mathbb{R}$-invariant tubular neighbourhood $\overline{U}$ of the $\mathbb{R}$-orbit of y. The group G acts on $\mathbb{R} \times B$ by the diagonal action. The immersion i is equivariant with respect to the natural projection $G \to \overline{\Gamma}$ and with respect to the action of $\mathbb{R}$ by translations.The immersion i induces an isomorphism from the orbifold quotient of $\mathbb{R} \times B$ by G to the orbifold quotient of $\overline{U}$ by $\overline{\Gamma}$ which is an invariant tubular neighbourhood of the $\mathbb{S}^1$-orbit of the projection $[y_0]$ of y_0 to $|Q|$. In particular G is the fundamental group of this tubular neighbourhood.

The quotient of $\mathbb{R} \times B$ by the subgroup generated by $(\gamma_0, 1)$ is isomorphic to $\mathbb{S}^1 \times B$ and its quotient by G is isomorphic to the quotient $U = \mathbb{S}^1 \times_I B$ of $\mathbb{S}^1 \times B$ by the diagonal action of the group I quotient of G by the subgroup generated by $(\gamma_0, 1)$. This describes an orbifold $\mathbb{S}^1$-invariant tubular neighbourhood of the $\mathbb{S}^1$-orbit of $[y_0]$. The action of I on B is effective because the action of $\mathbb{S}^1$ on Q is assumed to be effective. □

3.4 The energy function

The energy function E is well defined on $\Omega_{x,y}$ and Ω_X. As it is invariant by the action of $\mathcal{G}$, it gives a well defined function on $|\Lambda Q|$ still noted E. We list below some of its properties, refering for instance to Klingenberg [11,12] for the proofs.

3.4.1. E is a differentiable function on Ω_X or $\Omega_{x,y}$. The gradient grad E of E at $[c]_x$ is given by the formula:

$$\text{grad } E(v) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle \dot{c}_i(t), \nabla v_i(t) \rangle dt,$$

where $v = (v_1, \ldots, v_k) \in H^1(c^* TX)$.

The critical points of E on Ω_X (resp. $\Omega_{x,y}$) are in one to one correspondence with the equivalence classes of based closed geodesic $\mathcal{G}$-paths (resp. of geodesic $\mathcal{G}$-paths from x to y).

3.4.2. The Palais-Smale condition (C). For a compact orbifold (resp. a complete orbifold) the Palais-Smale condition (C) holds for the function E on $|\Lambda Q|$ (resp. on $\Omega_{x,y}$).

Namely, let c_m be a sequence of closed $\mathcal{G}$-paths (resp. of $\mathcal{G}$-paths from x to y) of class H^1 such that

(i) the sequence $E(c_m)$ is bounded,

(ii) the sequence $|\text{grad } E(c_m)|$ tends to zero;

then the sequence $[c_m]$ (resp. $[c_m]_{x,y}$) has accumulations points and any converging subsequence converges to a geodesic.

This implies that, for $a \geq 0$, the set of critical points of E in the subspaces $E^{-1}([0, a])$ of $|\Lambda Q|$ or $\Omega_{x,y}$ is compact.

3.4.3. The gradient flow. The vector field $-$grad E generates a local flow ϕ_t on Ω_X which commutes with the action of $\mathcal{G}$. On the quotient $|\Lambda Q| = \mathcal{G} \backslash \Omega_X$ it gives a local flow which is defined for all $t \geq 0$ if Q is compact.

Similarly the vector field $-$grad E on $\Omega_{x,y}$ generates a local flow ϕ_t which is defined for all $t \geq 0$ when Q is complete.

On Ω_X, when Q is compact, the local flow ϕ_t is $\overline{\mathcal{G}}$-complete in the following sense,
where $\overline{\mathcal{G}} := \mathcal{G} \times_X \Omega_X$ is the groupoid of germs of change of charts of the orbifold

ΛQ. Given $\overline{x} \in \Omega_X$ and $\tau \geq 0$, one can find a $\overline{\mathcal{G}}$-path $\overline{c} = (\overline{g}_0, \overline{c}_1, \overline{g}_1, \ldots, \overline{c}_k, \overline{g}_k)$ over a subdivision $0 = t_0 \leq t_1 \leq \cdots \leq t_k = \tau$ of the interval $[0, \tau]$ such that $\omega(\overline{g}_0) = \overline{x}$ and $\overline{c}_i(t) = \phi_{t-t_{i-1}}(\overline{c}_i(t_{i-1}))$ for $i = 1, \ldots, k$. The image of this path in $|\Lambda Q|$ is the ϕ_t trajectory of the projection of $\overline{x}$ for $t \in [0, \tau]$. When $\overline{x}$ remains in a small neighbourhood, such a $\overline{\mathcal{G}}$-path exists over the same subdivision and varies continuously.

Given another such $\overline{\mathcal{G}}$-path $\overline{c}' = (\overline{g}'_0, \overline{c}'_1, \overline{g}'_1, \ldots, \overline{c}'_k, \overline{g}'_k)$ issuing from $\overline{x}'$ defined over the same subdivision of the interval $[0, \tau]$ and an element $\overline{g} \in \overline{\mathcal{G}}'$ with source $\overline{x}$ and target $\overline{x}'$, then there are unique continuous maps $h_i : [t_{i-1}, t_i] \to \overline{\mathcal{G}}$ such that $\alpha(h_i(t)) = \overline{c}_i(t)$, $\omega(h_i(t)) = \overline{c}'_i(t)$, $\overline{g}'_0 h_1(0) = \overline{g}\overline{g}_0$ and $h_i(t_i)\overline{g}_i = \overline{g}'_i h_{i+1}(t_i)$ for $i = 1, \ldots, k-1$. This implies easily the following

3.4.4. Lemma. *: Given a "map" f from a compact space K to ΛQ (i.e. a morphism from K to $\overline{\mathcal{G}}$) and $\tau \geq 0$, there is a unique homotopy f_t of f parametrized by $t \in [0, \tau]$ whose projection to $|\Lambda Q|$ is the flow ϕ_t applied to the projection of f.*

.

3.4.5 Definition of ϕ-families. Let P be either $|\Lambda Q|$ or $\Omega_{x,y}$. In the first case we assume that $|Q|$ compact and in the second case that $|Q|$ is complete. For a number $a \in \mathbb{R}$, we denote P^a the set of point of P for which the value of the energy function is $\leq a$.

A ϕ-family (see [12], p. 20) is a collection $\mathcal{F}$ of non-empty subsets F of P such that E is bounded on each F and for $F \in \mathcal{F}$, then $\phi_t(F) \in \mathcal{F}$ for all $t > 0$.

Let $a \in \mathbb{R}$ and assume that there is $\epsilon > 0$ such that E has no critical values in $]a + \epsilon]$. A ϕ-family of P mod P^a if a ϕ-family $\mathcal{F}$ such that each member F is not contained in $P^{a+\epsilon}$. Note that a ϕ-family is always a ϕ-family of P mod P^a for $a < 0$.

The proof of the following theorem is similar to the proof of the corresponding theorem in the classical case (see for instance [12], p. 21). Up to the end of this section, we assume that Q is compact (resp. complete) if $P = |\Lambda Q|$ (resp. $P = \Omega_{x,y}$).

3.4.6. Theorem. *The critical value $a_{\mathcal{F}}$ of a ϕ-family $\mathcal{F}$ of P mod P^a, i.e. the number defined by*

$$a_{\mathcal{F}} = \inf_{F \in \mathcal{F}} \sup E|F,$$

is always $> a$ and there is a critical point of E with value $a_{\mathcal{F}}$.

Applying this theorem to the ϕ-family formed by the points of a connected component of P and for $a < 0$, we get the following corollary.

3.4.7. Corollary. *The energy function E restricted to a connected component of P assumes its infimum in some point, and such a point is a critical point of E.*

3.4.8. General remarks. Let c be a $\mathcal{G}$-geodesic path from x to y (respectively a closed $\mathcal{G}$-geodesic path based at x). The usual expression of the Hessian of the energy function at $T_{[c]_{x,y}}\Omega_{x,y}$ (resp. at $T_{[c]_x}\Omega_x$) is established as in the classical case. The index and nullity of $[c]_{x,y}$ (resp. of $[c]_x$) are defined as usual. The theory of Bott concerning the index and nullity of the m-th iterate of closed geodesics (see [5, p.495-98] and [11]) extends to the case of orbifolds. Anyway we shall show in the next section that we can consider only the developable case and, in that case, we can apply the results of Grove-Tanaka [6].

4. Geometric equivalence of closed geodesics

4.1. Geometric equivalence classes

In this section we describe, for a given closed geodesic of positive length on an orbifold Q, the set of closed geodesics on Q which are geometrically equivalent to it, i.e. those closed geodesics which have the same projection to $|Q|$. We shall also describe small invariant tubular neighbourhoods of their $\mathbb{S}^1$-orbits. To each geometric equivalence class of closed geodesics of positive length we associate a group which is an extension of $\mathbb{Z}$ by a finite subgroup. The elements of this group which are not in this finite subgroup correspond bijectively to representative in Ω_X of elements in this class which are closed $\mathcal{G}$-geodesic paths with an initial vector proportional to a given unit vector ξ. In the classical case, this groupis $\mathbb{Z}$ and its non zero elements correspond to the multiples of a primitive closed geodesic.

Then we prove that the study of the $\mathbb{S}^1$-invariant tubular neighbourhoods in $\Lambda(Q)$ of the closed geodesics in a geometric equivalence class is equivalent to the corresponding study in the developable caae.

4.1.1. The group attached to a geometric equivalence class.

Let $x \in X$ and let (E_x, e_x) be as in 2.4.4 the pointed principal $\mathcal{G}$-bundle over T_xX representing the exponential morphism based at x. Let $\xi \in T_xX$ be a unit vector and let (E_ξ, e_ξ) be the pointed bundle over $\mathbb{R}$ pull back of (E_x, e_x) by the map $t \mapsto t\xi$. To simplify the notations, we note p the projection $E_\xi \to \mathbb{R}$ and $\alpha : E_\xi \to X$ the action map. For a local section $s : [\tau - \epsilon, \tau + \epsilon] \to E_\xi$ such that $s(\tau) = e$, we note $\alpha(\dot{e})$ the velocity vector of $\alpha \circ s$ at τ. The elements $e \in E_\xi$ with $p(e) = \tau \neq 0$ correspond bijectively to the equivalence classes of geodesic $\mathcal{G}$-paths with initial vector equal to $\tau\xi$. Such an element represents a closed geodesic $\mathcal{G}$-path if and only if $\alpha(\dot{e}) = \xi$.

The pointed principal $\mathcal{G}$-bundle (E_ξ, e_ξ) is characterized up to a unique isomorphism by the following property: For any local section s of p, the composition $\alpha \circ s$ is a geodesic arc in X with unit speed; moreover $\alpha(\dot{e}_\xi) = \xi$.

Let $e \in E_\xi$ be such that $p(e) = \tau$ and $\alpha(\dot{e}) = \xi'$. The pull back of (E_ξ, e) by the translation $t \mapsto t + \tau$ is canonically isomorphic to $(E_{\xi'}, e_{\xi'})$. If $\xi' = \xi$, this amounts to say that there is a unique homeomorphism $h : E_\xi \to E_\xi$ projecting to the translation $t \mapsto t + \tau$, commuting with the right action of $\mathcal{G}$ and sending e_ξ to e. Such a map will be called an automorphism of E_ξ; it is uniquely determined by e. If c is a closed $\mathcal{G}$-geodesic based at x corresponding to $e = h(e_\xi)$ and m is a non-zero integer, then c^m corresponds to $h^m(e_\xi)$.

Let H be the group of such automorphisms of E_ξ. It projects to a group of translations of $\mathbb{R}$ which is discrete because, for ϵ small enough, there is no closed $\mathcal{G}$-geodesic loop of positive length $< \epsilon$ with initial vector proportional to ξ. Assume that this group is not trivial (equivalently that there is a closed $\mathcal{G}$-geodesic loop of positive length with initial vector proportional to ξ); it is an infinite cyclic group generated by an element $\tau_0 > 0$. Let $\phi : H \to \mathbb{Z}$ be the homomorphism defined by the relation $p(h(e_\xi)) = \phi(h)\tau_0$. The kernel of ϕ is canonically isomorphic to $\mathcal{G}_\xi$, the finite group formed by the elements of $\mathcal{G}$ fixing ξ. The action of $g \in \mathcal{G}_\xi \subset H$ on an element $e \in E_\xi$ represented by a geodesic $\mathcal{G}$-path c is the element represented by the path $g.c$. We summarize those considerations.

4.1.2. Proposition. *Let $\xi \in T_xX$ be a unit vector. Assume that there is a closed*

is an exact sequence of groups

$$1 \to \mathcal{G}_\xi \to H \xrightarrow{\phi} \mathbb{Z} \to 0$$

where $\mathcal{G}_\xi$ is the group of elements of $\mathcal{G}$ whose differential fixes ξ. The group H is the group of automorphisms of the principal $\mathcal{G}$-bundle E_ξ over $\mathbb{R}$ corresponding to the full geodesic with initial vector ξ.

The elements of H which are not in the subgroup $\mathcal{G}_\xi$ correspond bijectively to the closed $\mathcal{G}$-geodesics based at x with non-zero initial vector proportional to ξ. Let c be a closed $\mathcal{G}$-geodesic path with minimal positive length τ_0 with initial vector $\tau_0\xi$ correponding to an element $h_0 \in H$. Then any closed $\mathcal{G}$-geodesic of positive length based at x with initial vector proportional to ξ corresponding to $h = gh_0^m \in H$, where $g \in \mathcal{G}_\xi$ and m is a non-zero integer, is represented by the $\mathcal{G}$- path $c_h = g.c^m$.

4.1.3. Construction of a developable model. Given a geodesic $\mathcal{G}$-path $c = (g_0, c_1, \ldots, c_k, g_k)$ from x to y, and a vector $v \in T_xX$, we can construct along each c_i a parallel vector field v_i such that $Dg_0(v_1(0)) = v,\ Dg_i(v_{i+1}(t_i)) = v_i(t_i)$. The map $T_xX \to T_yX$ mapping v to $Dg_k^{-1}v_k(1) \in T_yX$ will be called the parallel transport along c. It depends only on the equivalence class of c and maps the orthogonal to the initial vector of c to the orthogonal to the terminal vector of c. In particular if c is a closed $\mathcal{G}$-geodesic path based at x with initial vector proportional to ξ, it maps to itself the subspace orthogonal to ξ.

Let N_0 be the subspace of T_xX orthogonal to ξ. Given $h \in H$, let $\rho(h)$ be the isometry of N_0 given by the inverse of the parallel transport along a closed geodesic $\mathcal{G}$-loop corresponding to $h(e_\xi)$. The map $h \mapsto \rho(h)$ is a homomorphism $\rho : H \to \mathrm{I}som\ N_0$. Let N_0^ϵ be the ball of radius ϵ centered at the origin of N_0, and let $N^\epsilon = \mathbb{R} \times N_0^\epsilon$. The group H acts properly on N^ϵ by the formula

$$h(t, \nu) = (t + \phi(h)\tau_0, \rho(h)\nu).$$

Let $\tilde{Q}$ be the orbifold $H\backslash N^\epsilon$. We note $\tilde{a}$ the map $\mathbb{R} \to N^\epsilon = \mathbb{R} \times N_0^\epsilon$ sending t to $(t, 0)$, $\tilde{x}$ the point $\tilde{a}(0)$ and $\tilde{\xi}$ the velocity vector of $\tilde{a}$ at $t = 0$.

4.1.4. Theorem. *For ϵ small enough, one can define on N^ϵ a Riemannian metric invariant by H and such that $\tilde{a}$ is a geodesic with speed one, and an open Riemannian immersion u from the orbifold $\tilde{Q} = H\backslash N^\epsilon$ to the orbifold $Q = \mathcal{G}\backslash X$. The immersion u induces a bijection between the set of geodesic $H \ltimes N^\epsilon$-loops based at $\tilde{x}$ with initial vector a non-zero multiple of $\tilde{\xi}$ and the set of geodesic $\mathcal{G}$-loops based at x with initial vector a non-zero multiple of ξ. Moreover the immersion u induces isomorphisms of the orbifold neighbourhoods of the corresponding closed geodesics and of their $\mathbb{S}^1$-orbits.*

Proof. Let h_0 be an element of H such that $\phi(h_0) = 1$ and let $c = (g_0, c_1, \ldots, c_k, g_k)$ be a closed geodesic $\mathcal{G}$-path over a subdivision $(0 = t_0 < t_1 < \cdots < t_k = 1)$ with initial vector $\tau_0\xi$ whose equivalence class in Ω_x corresponds to h_0. We can choose c in its equivalence class so that $g_0 = g_k = 1_x$ and that there is an $\epsilon > 0$ such that, for each $t \in [t_{i-1}, t_i]$, the point $c_i(t)$ is the center of a convex geodesic ball $B(c_i(t), 2\epsilon)$ of radius 2ϵ. We can also assume that the length of each c_i is smaller than ϵ. Let δ be a positive number smaller than the numbers $(t_{i-1} - t_i)/2$. Each c_i can be extended to a geodesic segment $\bar{c}_i :]t_{i-1} - \delta, t_i + \delta[\to X$.

For each $i = 1, \ldots, k$, let $U_i =]\tau_0(t_{i-1} - \delta), \tau_0(t_i + \delta)[\times N_0^\epsilon \subset \mathbb{R} \times N_0^\epsilon = N^\epsilon$ and let $\tilde{X}$ be the disjoint union of the U_i, namely the union of the $\tilde{U}_i := (\{i\}, U_i)$. The composition of the natural projection $\tilde{u}_i : \tilde{U}_i \to U_i$ with the projection $N^\epsilon \to H\backslash N^\epsilon = \tilde{Q}$ gives an atlas of uniformizing chart for $\tilde{Q}$. The pseudogroup $\tilde{\mathcal{P}}$ of change of charts of this atlas is generated by the following elements (whenever defined):

i) $\tilde{u}_i^{-1}\tilde{u}_{i+1}$ for $i = 1, \ldots, k-1$,

ii) $\tilde{h}_0 := \tilde{u}_k^{-1} h_0 \tilde{u}_1$,

iii) for each $d \in \mathcal{G}_\xi$ and $i = 1, \ldots, k$, the maps $\tilde{d}_i : \tilde{U}_i \to \tilde{U}_i$ sending $(i, (\tau_0 t, \nu))$ to $(i, (\tau_0 t, \rho(d)\nu))$.

To describe $\tilde{Q}$, we can replace N^ϵ by $\tilde{X}$ and the groupoid $H \ltimes N^\epsilon$ by the groupoid $\tilde{\mathcal{G}}$ of germs of $\tilde{\mathcal{P}}$. The immersion $\tilde{Q} \to Q$ will be defined as a continuous functor $u : \tilde{\mathcal{G}} \to \mathcal{G}$ that we are going to construct below.

Given $\nu \in N_0^\epsilon$, let $\overline{\nu}_i$ be the parallel vector fields along the $\overline{c}_i$ such that $\overline{\nu}_1(0) = \nu$ and $\overline{\nu}_i(t_i) = (Dg_i)(\overline{\nu}_{i+1}(t_i))$. Let $u_i : \tilde{U}_i \to X$ be the open embedding mapping $(i, (\tau_0 t, \nu))$ to $exp_{c_i(t)}\overline{\nu}_i(t)$. The functor u restricted to the space of units will be the open immersion $u : \tilde{X} \to X$ equal to u_i on $\tilde{U}_i$.

We note $\mathcal{P}$ the pseudogroup of local isometries of X whose groupoid of germs is $\mathcal{G}$. It contains in particular, for each $i = 1, \ldots, k-1$, the isometry $\overline{g}_i : B(c_{i+1}(t_i), 2\epsilon) \to B(c_i(t_i), 2\epsilon)$ whose germ at $c_{i+1}(t_i)$ is g_i (see 2.1.5). It also contains, for each $i = 1, \ldots, k$ and $d \in \mathcal{G}_\xi$, the isometry $\overline{d}_i$ of the ball $B(c_i(t_{i-1}), 2\epsilon)$, leaving invariant c_i, the germ at x of $\overline{d}_1$ being d and the germ of $\overline{d}_i$ at $c_i(t_{i-1})$ being the germ of $\overline{g}_{i-1}^{-1}\overline{d}_{i-1}\overline{g}_{i-1}$. We have the following relations, whenever both sides are defined:

i') $u_i(\tilde{u}_i^{-1}\tilde{u}_{i+1})u_{i+1}^{-1} = \overline{g}_i$ for $i = 1, \ldots, k-1$,

ii') $u_k \tilde{h}_0 = u_1$,

iii') $u_i \tilde{d}_i u_i^{-1} = \overline{d}_i$ for each $d \in \mathcal{G}_\xi$ and $i = 1, \ldots, k$.

Therefore we can define the functor $u : \tilde{\mathcal{G}} \to \mathcal{G}$ by associating respectively to the germs of the $\tilde{u}_i^{-1}\tilde{u}_{i+1}$, of $\tilde{h}_0$ and of the $\tilde{d}_i$, respectively the germs of the $\overline{g}_i$, of the identity and of $\overline{d}_i$ at the points corresponding to each other by $u : \tilde{X} \to X$.

The above relations show that the Riemannian metric on $\tilde{X}$ which is the pull back of the Riemannian metric on X by the immersion $\tilde{X} \to X$ is $\tilde{\mathcal{G}}$-invariant. Hence we obtain on N^ϵ an H-invariant Riemannian metric for which $\tilde{a}$ is a geodesic.

Instead of looking at the closed geodesic $(H \ltimes N^\epsilon)$-loops based at $\tilde{x}$ with initial vector proportional to $\tilde{\xi}$, we can equivalently consider the closed geodesic $\tilde{\mathcal{G}}$-loops based at $\tilde{x} = (1, \tilde{x}) \in \tilde{U}_1$ with initial vector proportional to $\tilde{\xi} = (1, \tilde{\xi}) \in T_{\tilde{x}}\tilde{U}_1$. For $d \in \mathcal{G}_\xi$, let $\tilde{d}$ be the element of $\tilde{\mathcal{G}}_{\tilde{\xi}}$ mapped to d by u.

Let $\tilde{c}_{h_0} = (\tilde{g}_0, \tilde{c}_1, \ldots, \tilde{c}_k, \tilde{g}_k)$ be the closed geodesic $\tilde{\mathcal{G}}$-path over the subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ with initial vector $\tau_0\tilde{\xi}$, where $\tilde{c}_i : [t_{i-1}, t_i] \to \tilde{U}_i$ is such that $u_i\tilde{c}_i = c_i$, $\tilde{g}_0 = 1_{\tilde{x}}$, $\tilde{g}_k$ is the germ at $\tilde{x}$ of $\tilde{h}_0$, and for $i = 1, \ldots, k-1$, $\tilde{g}_i$ is the germ of $\tilde{u}_i^{-1}\tilde{u}_{i+1}$ at $\tilde{c}_{i+1}$. The functor $u : \tilde{\mathcal{G}} \to \mathcal{G}$ maps $\tilde{c}_{h_0}$ to c_{h_0}, and more generally, for $h = dh_0^m$, where $d \in \mathcal{G}_\xi$, it maps $\tilde{c}_h := \tilde{d}.\tilde{c}_{h_0}^m$ to $c_h := d.c_{h_0}^m$. The immersion u induces an immersion $\Omega_u : \Omega_{\tilde{X}}(\tilde{\mathcal{G}}) \to \Omega_X(\mathcal{G})$. It is clear that Ω_u maps equivariantly small neighbourhoods of $[\tilde{c}_h]_{\tilde{x}}$ to small neighbourhoods of $[c_h]_x$ for every $h \in H$. Moreover the map $\Lambda_u : \Lambda(\tilde{Q}) \to \Lambda(Q)$ induced by Ω_u gives $\mathbb{S}^1$-invariant isomorphisms of tubular neighbourhoods of the $\mathbb{S}^1$-orbits of $[\tilde{c}_h]$ and $[c_h]$. $\square$

4.1.5. Description of a $\mathbb{S}^1$-invariant tubular neighbourhood of the $\mathbb{S}^1$-orbit

of $[c_h]$**.** To describe such a tubular neighbourhood, using the preceding theorem, we can as well describe a $\mathbb{S}^1$-invariant tubular neighbourhood of the $\mathbb{S}^1$-orbit of the corresponding based closed $(H \ltimes N^\epsilon)$-geodesic $\tilde{c}_h = (\tilde{a}^{m\tau_0}, h)$, where $m = \phi(h)$ and $\tilde{a}^{m\tau_0}(t) = \tilde{a}(m\tau_0 t) = (m\tau_0 t, 0) \in \mathbb{R} \times N_0^\epsilon = N^\epsilon$.

The subgroup of H leaving invariant the $\mathbb{R}$-orbit of $\tilde{c}_h$ is the centralizer Z_h of h in H. The image of Z_h by ϕ is a subgroup of $\mathbb{Z}$ containing $\phi(h) = m$, hence it is generated by a positive integer d dividing m. Comparing with the general situation described in 3.3.7, Y corresponds to the connected component of Ω_{N^ϵ} containing $[\tilde{c}_h]_{\tilde{x}}$, the group $\overline{\Gamma}$ corresponds to Z_h (it is equal to G in our case), and the homomorphism $\psi : G \to \mathbb{Z}\ 1/r$ to the homomorphism $Z_h \to \mathbb{Z}\ 1/r$, where $r = m/d$, mapping $h' \in Z_h$ to $\phi(h')/r$. Also the element γ_0 corresponds to $h \in Z_h$, and the group I to the quotient I_h of Z_h by the subgroup generated by h.

Let $\mathcal{N}_0$ be the subspace of the tangent space of Ω_{N^ϵ} at $[\tilde{c}_h]_{\tilde{x}}$ orthogonal to the speed vector field along $\tilde{c}_h$. The elements of $\mathcal{N}_0$ are the H^1-vector fields $t \mapsto v(t)$ along the geodesic $\tilde{a}^{m\tau_0}$ such that the image of $v(t)$ by the differential of h is $v(t+1)$ and $\int_0^1 \langle v(t), \dot{\tilde{a}^{m\tau_0}}(t)\rangle dt = 0$. The group Z_h acts by isometries on $\mathcal{N}_0$ through the differential of its elements acting on N^ϵ, because those elements commute with h. Indeed let us note v' the vector field along $\tilde{a}^{m\tau_0}$ associating to t the image $v'(t)$ of $v(t - \psi(h'))$ by the differential Dh' of $h' \in Z_h$. We have $Dh[v'(t)] = DhDh'[v(t - \psi(h'))] = Dh'[v(t + 1 - \psi(h'))] = v'(t + 1)$. As h itself acts trivialy, this gives an action on $\mathcal{N}_0$ of the quotient I_h of Z_h by the subgroup generated by h. This induces an action of I_h by isometries on the image $\mathcal{D}_0$ in Ω_X by the exponential map of a small ball in $\mathcal{N}_0$ centered at 0.

The homomorphism $\psi : Z_h \to \mathbb{Z}\ 1/r$ induces a homomorphism $\phi_h : I_h \to \mathbb{S}^1$ whose image is a subgroup of order $r = m/d$. The quotient $\mathbb{S}^1 \times_{I_h} \mathcal{D}_0$ of $\mathbb{S}^1 \times \mathcal{D}_0$ by the diagonal action of I_h is isomorphic as an orbifold (see 3.3.8) to an $\mathbb{S}^1$-invariant tubular neighbourhood U of $\mathbb{S}^1.[\tilde{c}_h]$, or equivalently of $\mathbb{S}^1.[c_h]$.

4.2. Local homological invariants in a geometric equivalence class

Following Gromoll-Meyer [5] and Grove-Tanaka [6] we define local homological invariants for isolated critical orbits and compare them for geodesics in a geometric equivalence class.

4.2.1.**Local homological invariants for isolated critical orbits.** Let c be a closed geodesic $\mathcal{G}$-path of positive length based at $x \in X$. We have just seen that a $\mathbb{S}^1$-invariant tubular neighbourhood of the $\mathbb{S}^1$-orbit of the equivalence class $[c]$ of c is isomorphic to the quotient $\mathbb{S}^1 \times_I \mathcal{D}_0$ of $\mathbb{S}^1 \times \mathcal{D}_0$ by the diagonal action of a finite group I commuting with the action of $\mathbb{S}^1$. The fiber $\mathcal{D}_0$ is isomorphic to the image by the exponential map of a small ball in the subspace of $T_{[c]_x}\Omega_X$ orthogonal to the velocity vector field of c. We note E the energy function on $\mathbb{S}^1 \times \mathcal{D}_0$ (it is $\mathbb{S}^1$-invariant). We assume that $\mathbb{S}^1.[c]$ is an isolated critical orbit and, following Gromoll and Meyer [5], we want to define a local homological invariant $\mathcal{H}_*(\mathbb{S}^1.[c])$ depending only of the germ of neighbourhood of the orbit $\mathbb{S}^1.[c]$.

The restriction E_0 of the energy function to the fiber $\mathcal{D}_0$ has $[c]_x$ as an isolated critical point with critical value a. By taking $\mathcal{D}_0$ small enough, we can assume that $[c]_x$ is the only critical point of E_0. As in [4] (see also [5, p.502]), one can construct an admissible region W_0 for the function E_0; it is constained in $E_0^{-1}[a - \delta, a + \delta]$, where δ is a small positive number, and can be choosen to be invariant under the action of I. The local invariant $\mathcal{H}_*(E_0, [c]_x)$ is defined in [4] as the singular

homology $H_*(W_0, W_0^-)$ with rational coefficient, where W_0^- is the intersection of W_0 with $E_0^{-1}(a-\delta)$. Setting $W = \mathbb{S}^1 \times_I W_0$ and $W^- = \mathbb{S}^1 \times_I W_0^-$, the local homological invariant for the $\mathbb{S}^1$-orbit of $[c]$ is the finite dimensionla vector space defined by

$$\mathcal{H}_*(\mathbb{S}^1.[c]) := H_*(W, W^-).$$

It is isomorphic to the subspace $H_*(\mathbb{S}^1 \times W_0)^I$ of $H_*(\mathbb{S}^1 \times W_0)$ of elements left invariant by the induced action of I on the homology.

The dimension of $\mathcal{H}_k(\mathbb{S}^1.c)$ is noted $B_k([c])$ and is called the k-th local Betti number of the isolated orbit $\mathbb{S}^1.[c]$.

We have the analogue of Lemma 4 in [5] with the same proof. For $b \geq 0$, we note $|\Lambda Q|^b$ the subspace of $|\Lambda Q|$ which is the inverse image of $[0, b]$ by the energy function $E : |\Lambda Q| \to \mathbb{R}$.

4.2.2. Lemma. *Let Q be a compact Riemannian orbifold. Let $a > 0$ be the only critical value of the energy function $E : |\Lambda Q| \to \mathbb{R}^+$ in the interval $[a-\epsilon, a+\epsilon]$. Assume that the critical set in $E^{-1}(a)$ consists of finitely many critical orbits $\mathbb{S}^1.[c^1], \ldots \mathbb{S}^1.[c^r]$. Then*

$$H_*(|\Lambda Q|^{a+\epsilon}, |\Lambda Q|^{a-\epsilon}) = \sum_{i=1}^{r} \mathcal{H}_*(\mathbb{S}^1.[c_i]).$$

In this statement, the space $|\Lambda Q|$ can be replaced by a union of some of its connected components.

We have the analogue of Corollary 2 of Gromoll-Meyer [5] (see also [6]).

4.2.3. Theorem. *Let $\mathcal{C}$ be a geometric equivalence class of closed geodesics of positive length on the orbifold $Q = \mathcal{G}\backslash X$. We assume that each $\mathbb{S}^1$-orbit in this equivalence class is an isolated critical manifold for the energy function.*

1) There is a constant B such that $B_k([c]) < B$ for every $k > 2dim\ |Q|$ and every $[c] \in \mathcal{C}$.

2) There is a constant C such that the number of orbits $\mathbb{S}^1.[c]$, for $[c] \in \mathcal{C}$, such that $B_k([c]) \neq 0$ for $k > 2\ dim\ |Q|$, is bounded by C.

Proof. Although a more direct proof could be obtained following the arguments of Gromoll-Meyer and Grove-Tanaka, we shall deduce the theorem from the results of Grove-Tanaka [6].

Let $\xi \in T_x X$ be a unit tangent vector such that the elements of $\mathcal{C}$ are the $\mathbb{S}^1$-orbits of elements represented by closed geodesic $\mathcal{G}$-paths with initial vector proportional to ξ. Let N_0^ϵ be an ϵ-ball in the orthogonal to ξ in $T_x X$ centered at 0. According to 4.1, there is a group H acting properly by isometries on a Riemannian manifold diffeomorphic to $N^\epsilon = \mathbb{R} \times N_0^\epsilon$. The curve $\tilde{a} : t \mapsto (t, 0)$ is a geodesic line with unit speed left invariant by H. The elements of $\mathcal{C}$ correspond bijectively to the elements of the geometric class $\tilde{\mathcal{C}}$ of closed geodesics on the orbifold $\tilde{Q} = H\backslash N^\epsilon$ having the same image in $|\tilde{Q}|$ as $\tilde{a}$ and the corresponding elements have the same local homological invariants. Therefore it is sufficient to prove the theorem for $\tilde{\mathcal{C}}$.

We have a surjective homomorphism $\phi : H \to \mathbb{Z}$; the kernel of ϕ is the finite subgroup of H fixing the velocity vector $\dot{\tilde{a}}(0) = \tilde{\xi}$. Recall that the $\mathbb{S}^1$-orbits of the elements of $\mathcal{C}$ are parametrized by the conjugacy classes of the elements of H whose image by ϕ is non zero. To each element h of H is associated the element of

$\tilde{\mathcal{C}}$ represented by the geodesic path $\tilde{c}_h = (\tilde{a}^{\phi(h)\tau_0}, h)$, where τ_0 is the minimal length of the closed geodesics in the class $\mathcal{C}$.

As H is an extension of $\mathbb{Z}$ by a finite group, we can find an element $h_1 \in H$ in the center of H with $\phi(h_1) = n_1 > 0$. Let Γ be the quotient of H by the subgroup H_1 generated by h_1. It is a finite group acting on the quotient Riemannian manifold $M := H_1 \backslash N^\epsilon$. Let $\pi : H \to \Gamma$ be the quotient projection. The natural projection $N^\epsilon \to M$ induces an isomorphism $\tilde{Q} = H \backslash N^\epsilon \to \overline{Q} := \Gamma \backslash M$ of the quotient orbifolds. The composition of $\tilde{a} : \mathbb{R} \to N^\epsilon$ with the natural projection onto M is a closed geodesic $\overline{a} : \mathbb{R} \to M$. Therefore we can replace $\tilde{\mathcal{C}}$ by the geometric equivalence class $\overline{\mathcal{C}}$ of closed geodesics on $\overline{Q}$ which have the same image in $|\overline{Q}|$ as the image of $\overline{a}$. To each $h \in H$ is associated the closed based geodesic $(\Gamma \ltimes M)$-path with initial vector proportional to $\overline{\xi} := \dot{\overline{a}}(0)$ represented by $\overline{c}_h := (\overline{a}^{\phi(h)\tau_0}, \pi(h))$.

For each $\gamma \in \Gamma$, let $\mathcal{C}_\gamma$ be the set of γ-invariant geodesics (in the sense of Grove) in M with non-zero initial vector proportional to $\dot{\overline{a}}(0)$. These geodesics correspond bijectively to the closed geodesics $\overline{c}_h$ with $\pi(h) = \gamma$ and $\phi(h) \neq 0$. As $\overline{\mathcal{C}}$ is the finite union of the $\mathbb{S}^1$-orbits of the elements of $\mathcal{C}_\gamma$, it is sufficient to prove the analogue of the theorem where $\mathcal{C}$ is replaced by $\mathcal{C}_\gamma$.

In the notations of Grove-Tanaka, let $\Lambda(M, \gamma)$ be the space of γ-invariant H^1-curves on M. (In our notations this corresponds to a union of connected components of $\Omega_M(\Gamma \ltimes M)$.). The natural action of $\mathbb{R}$ on $\Lambda(M, \gamma)$ gives the action of the circle $\mathbb{R}/s\mathbb{Z}$ considered in [6], where s is the order of γ. The restriction of the energy function to $\Lambda(M, \gamma)$ is noted E^γ. Let $\mu : \Lambda(M, \gamma) \to |\Lambda\overline{Q}|$ be the composition of the inclusion $\Lambda(M, \gamma) \to \Omega_M(\Gamma \ltimes M)$ with the quotient map to $\Gamma \backslash \Omega_M(\Gamma \ltimes M) = |\Lambda\overline{Q}|$. The image of μ is an open set isomorphic to the quotient of $\Lambda(M, \gamma)$ by the action of the centralizer Z_γ of γ in Γ. The map μ sends $\mathbb{R}/s\mathbb{Z}$-orbits to $\mathbb{S}^1$-orbits. In a small $\mathbb{S}^1$-invariant tubular neighbourhood of $[\overline{c}_h]$, where $h \in \pi^{-1}(\gamma)$, following Gromoll-Meyer [5], we have constructed in 4.2.1 an admissible region W and defined $B_k([\overline{c}_h])$ as the dimension of $H_k(W, W^-)$. The inverse image $\mu^{-1}(W)$ of W is a $\mathbb{R}/s\mathbb{Z}$-invariant admissible region and dim $H_k(\mu^{-1}(W), \mu^{-1}(W^-))$ is the local Betti number defined in Grove-Tanaka [6, p. 44] noted there $B_k(\overline{a}^{\phi(h)\tau_0}, \gamma)$. As $W = Z_\gamma \backslash \mu^{-1}(W)$, the vector space $H_k(W, W^-)$ is isomorphic to the vector subspace of $H_k(\mu^{-1}(W), \mu^{-1}(W^-))$ left invariant by Z_γ. Therefore $B_k([\overline{c}_h]) \leq B_k(\overline{a}^{\phi(h)\tau_0}, \gamma)$, and we can use the results of [6] where the existence of constants like B and C are established for $B_k(\overline{c}_h, \gamma)$, where $\overline{c}_h \in \mathcal{C}_\gamma$. □

5. Existence of geodesics

5.1. Existence of at least one closed geodesic of positive length

5.1.1. Theorem. *Let Q be a compact connected Riemannian orbifold. There exists at least one closed geodesic on Q of positive length in the following cases:*

a) Q is not developable,

b) the fundamental group of Q has an element of infinite order or is finite.

Proof. a) If $Q = \mathcal{G} \backslash X$ is not developable, there is a point $x \in X$ and a non trivial element g in the group $\mathcal{G}_x$ of elements of $\mathcal{G}$ fixing x such that the closed loop based at x represented by the $c = (1_x, c_1, g)$, where $c_1 : [0, 1] \to X$ is the constant map to x, is homotopically trivial (see for instance [2]). Hence in $|\Lambda Q|$ there is a continuous path joining the point z of $|\Lambda^0 Q|$ represented by c to a point z' of $|\Lambda^0 Q|$ represented by a constant loop. These two points are in distinct components of $|\Lambda^0 Q|$, and the

components are all compact. Moreover the sets of point $|\Lambda^\epsilon Q|$ of $|\Lambda Q|$ for which the energy function is smaller than ϵ for various $\epsilon > 0$ form a fundamental system of neighbourhoods of $|\Lambda^0 Q|$ (to see this adapt the proof given in [11, pages 30-31], using 2.4.6). Therefore the family of paths in $|\Lambda Q|$ joining z to z' is a ϕ-family mod $|\Lambda^0 Q|$ and we can apply 3.4.6.

b) The energy function restricted to the connected component of $|\Lambda Q|$ corresponding to an element of infinite order of the fundamental group of Q attains its infimum (cf. 3.4.7) at some point; this point is necessarily of positive length, as this curve represents an element of infinite order. In the case where the fundamental group of Q is finite, then either Q is not developable and we can apply a), or its universal covering is a compact Riemannian manifold M. Then the classical result of Fet implies the existence of a closed geodesic on M of positive length and its projection gives a closed geodesic on Q of positive length. $\square$

5.1.2. Remark. For the existence of a closed geodesic of positive length on compact orbifolds, the only case left open by the preceding theorem would be the following one. Let Γ be an infinite group all of whose elements are of finite order acting properly on a Riemannian manifold M by isometries with compact quotient $Q = \Gamma\backslash M$; does there exists a non constant geodesic $c : [0,1] \to M$ and an element $\gamma \in \Gamma$ such that the differential of γ maps $\dot{c}(0)$ to $\dot{c}(1)$?. Note that such a group Γ would be finitely presented, and no examples of such groups are known yet.

5.1.3. Remark. The argument used in a) applies whenever two distinct connected components of $|\Lambda^0 Q|$ are contained in the same connected component of $|\Lambda Q|$.

For instance, consider on $\mathbb{S}^2$ a Riemannian metric invariant by a rotation ρ of order n fixing the north pole N and the south pole S. The quotient of $\mathbb{S}^2$ by the group generated by ρ is a Riemannian orbifold. Let k be an integer not divisible by n; the elements of $|\Lambda Q|$ represented by the closed $\mathcal{G}$-paths (c, ρ^k), where $c : [0,1] \to \mathbb{S}^2$ is either the constant map to N or to S are in the same connected component of $|\Lambda Q|$, but in distinct components of $|\Lambda^0 Q|$. Therefore the argument in a) shows that there exists a geodesic segment $c : [0,1] \to \mathbb{S}^2$ of positive length such that the differential of ρ^k maps $\dot{c}(0)$ to $\dot{c}(1)$.

4.2. Existence of infinitely many geodesics

The following is the generalization of a theorem of Serre [16].

4.2.1. Theorem. *Let Q be a compact connected orbifold. Given two points $\overline{x}$ and $\overline{x}'$ of $|Q|$, there exist an infinity of geodesics from $\overline{x}$ to $\overline{x}'$.*

Proof. Let $Q = \mathcal{G}\backslash X$ and let x and x' be two points of X projecting to $\overline{x}$ and $\overline{x}'$. We have to prove that there exists an infinity of equivalence classes of geodesic $\mathcal{G}$-paths from x to x'. The connected components of $\Omega_{x,x'}$ correspond bijectively to the elements of the fundamental group of Q (i.e. of BQ, see 3.3.5). The energy function assumes its infimum on each connected component (see 3.4.7). Therefore it suffices to consider the case where the fundamental group of Q is finite and, after passing to the universal covering, the case where Q is simply connected. Then the fundamental rational class of Q (see 2.5.3) gives a non-trivial element in $H_n(BQ, \mathbb{Q})$, and by Serre [16], p. 484, the rational Betti numbers b_i of $\Omega_{x,x'}$, which has the same homotopy type as the loop space of BQ, do not vanish for an infinity of value of i. This would contradict the existence of only finitely many critical points of the energy function on $\Omega_{x,y}$ (see Gromoll-Meyer [4] or the more direct arguments of Seifert-Threlfall [15]). $\square$

We next prove the analogue of Gromoll-Meyer theorem (see also theorem 4.1 of Grove-Tanaka [6]) about the existence of infinitely many geometrically distinct closed geodesics.

5.2.2. Theorem. *Let $Q = \mathcal{G}\backslash X$ be a compact connected Riemannian orbifold. Assume there is only a finite number of geometric equivalence classes of closed geodesics on Q of positive length. There is a constant D such that the rational Betti numbers B_k of $|\Lambda Q|$ are bounded by D for $k \geq 2$ dim Q.*

Proof. As the proof of Theorem 4 of Gromoll-Meyer [5], or Theorem 4.1 in Grove-Tanaka [6] using 4.2.3. □

The next corollary follows from 5.2.2 and the Vigué-Sullivan theorem [18].

5.2.3. Corollary. *Let Q be a compact simply connected Riemannian orbifold. Assume that the rational cohomology of $|Q|$ is not generated by a single element. Then there are infinitely many geometric equivalence classes of closed geodesics of positive length on Q.*

Proof. As BQ is simply connected, $|Q|$ is also simply connected (the natural map $\pi_1(BQ) \to \pi_1(|Q|)$ is always surjective). Moreover the projection $BQ \to |Q|$ induces an isomorphism on the rational homology, so it is a rational homotopy equivalence. This implies that the map $\Lambda BQ \to \Lambda|Q|$ induced on the free loop spaces is also a rational homotopy equivalence. So $\Lambda|Q|$ has the same rational Betti numbers as ΛBQ

On the other hand the projection $B\Lambda Q \to |\Lambda Q|$ induces an isomorphism in rational cohomology by 2.5.3. As $B\Lambda^c Q$ is weakly homotopy equivalent to ΛBQ by 3.2.2 and to $B(\Lambda Q)$ by 3.3.5, it follows finally that $\Lambda|Q|$ has the same rational Betti numbers as $|\Lambda Q|$.

As $|Q|$ is simply connected, the rational Betti numbers of its free loop space $\Lambda|Q|$ are all finite by Serre [16]. If there would be only a finite number of geometric equivalence classes of closed geodesics on Q of positive length, then the rational Betti numbers of $|\Lambda Q|$ would be uniformly bounded by 5.2.2, so also those of $\Lambda|Q|$). By [18] this would imply that the rational cohomology of $|Q|$ would be generated by a single element, contrary to the hypothesis. □

The next theorem is the extension to the case of orbifolds of a result of Viktor Bangert and Nancy Hingston [1].

5.2.4. Theorem. *Let Q be a connected compact Riemannian orbifold of dimension > 1 whose fundamental group is infinite abelian. Then there is an infinity of geometrically distinct closed geodesics of positive length on Q.*

Proof. As mentioned in [1], the crucial case is when the fundamental group of Q is infinite cyclic. We can also assume Q orientable. We follow closely the arguments of [1]; as far as the homotopy properties are concerned, the role of the compact Riemannian manifold M is played by BQ. The fundamental class of Q gives a nontrivial element of the homology with rational coefficients of BQ in dimension > 1 (see 2.5.3), hence BQ has not the homotopy type of a circle and $\pi_n(BQ) \neq 0$ for some minimal $n > 1$. Let $z \in BQ$ be a base point. The connected components of $\Omega_z BQ$ correspond to the elements of the fundamental group of Q. For each integer $r > 0$ choose a loop l_r at z whose homotopy class if the r-th power t^r of a generator t of the fundamental group of Q.

The lemmas 1 and 2 of [1] provide the existence of a positive integer k such that, for all positive integer m, there exist elements $\alpha_m \in \pi_{n-1}(\Omega_z BQ, l_{mk})$ such that the image of α_m in $\pi_{n-1}(\Lambda BQ, l_{mk})$ is non trivial. The element α_m is described as follows. There is a map $f : (\mathbb{S}^{n-1}, *) \to (\Omega_z BQ, z)$ (where z is identified to the constant loop at z) so that the element α_m is represented by the map f_m obtained by composing with l_{mk} the loops in the image of f.

By 3.2.2 and 3.3.5, $\Omega_z BQ$ has the same weak homotopy type as Ω_x, where $x \in X$ and z have the same projection to $|Q|$. Let $[c_m]_x \in \Omega_x$ be a $\mathcal{G}$-loop based at x of minimal energy in the homotopy class of t^{mk}. Let $f' : (\mathbb{S}^{n-1}, *) \to (\Omega_x, [c_0]_x)$ be a map whose homotopy class corresponds to the homotopy class of f. Let $f'_m : (\mathbb{S}^{n-1}, *) \to (\Omega_x, [c_m]_x)$ mapping $y \in \mathbb{S}^{n-1}$ to the composition of $f'(y)$ with $[c_m]_x$. Considered as a morphism from $\mathbb{S}^{n-1}$ to the orbifold ΛQ, it is not homotopic to a constant.

Let κ_m be the infimum of the energy of the $\mathcal{G}$-loops in the homotopy class t^{km}. Let $F_m \subset |\Lambda Q|$ be the image of the composition of f'_m with the projection to $|\Lambda Q|$ and let τ_m be the critical value of the ϕ-family $\mathcal{F}$ (see 3.4.5) whose elements are the subsets $\phi_t(F_m)$ $(t > 0)$, where ϕ_t is the flow of $-\mathrm{grad}\, E$. Following [1] one can assume that $\kappa_m > \tau_m$, because otherwise either there would exist a continuous family of geometrically distinct closed geodesics with energy κ_m, or there would exist in the homotopy class of f'_m, by 3.4.4, a morphism whose image in $|\Lambda Q|$ would be contained in a $\mathbb{S}^1$-orbit; but this would contradict the condition that f'_m is not homotopic in ΛQ to a constant. The final part of the proof is as in [1]. □

To end up we reformulate in our framework some results of Grove and Halperin [7].

5.2.5. Theorem of Grove and Halperin. *Let Q be an orbifold quotient of a closed simply connected Riemannian manifold X by a finite group of isometries.*

1) If the dimension of Q is odd, there is a closed geodesic of positive length on Q in each connected component of ΛQ.

2) Assume that the rational homotopy groups $\pi_k(X)$ of X are non zero for an infinity of k. Then in each component of ΛQ, there exist an infinity of geometrically distinct closed geodesics on Q of positive length.

Proof. This theorem follows from Therorem A and B in [7, p. 173] using the remarks at the end of 3.3.6. □

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Université de Genève, Section de Mathématiques, 2-4 Rue du Lièvre, 1211 Genève 24